How to make the Landau Framework for Topological Order: Introduction to the recipe book

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Based on:

1. G.Y.C. and T. Pandey, J. Stat. Mech. (2017)

2. G.Y.C., arXiv:1710.04716

Talk at McGill University, October 12, 2017

<u>Outline:</u>

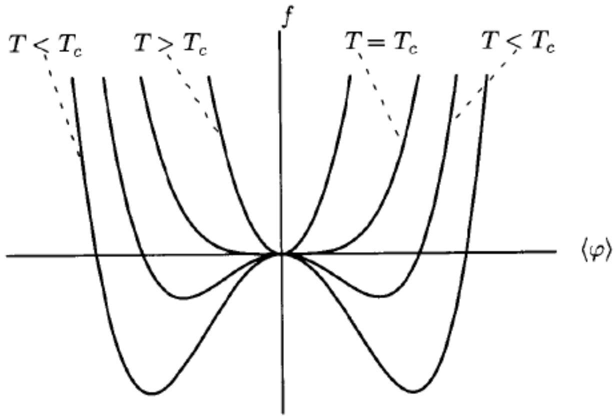
- ***Motivation: Landau Theory;***
  Phase Transitions without Local Order Parameters -- Examples

- ***Free Kitaev chain***
- ***Dimerized XY chain***
- ***Interacting dimerized Kitaev chain***
  <u>Exact results</u>
- ***Dimerized Two-Leg Ladders: Mean-Field (Free-fermionic) Approximation***
  Local vs Topological String Order Parameters; Winding Numbers (Topological indices)

- ***Conclusions & Future work***

Landau Framework (Paradigm):

E.g. Chaikin, Lubensky

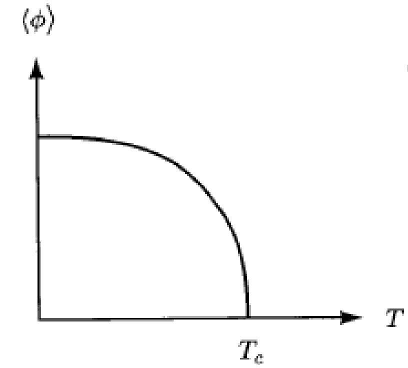


$$f(T, \langle\phi\rangle) = \frac{1}{2} r \langle\phi\rangle^2 + u \langle\phi\rangle^4, \qquad r = a(T - T_c).$$

$$\langle\phi\rangle = \begin{cases} 0 & \text{if } T > T_c; \\ \pm(-r/4u)^{1/2} & \text{if } T < T_c. \end{cases}$$

$$\langle\phi\rangle \sim (T_c - T)^\beta , \qquad \beta = 1/2 .$$

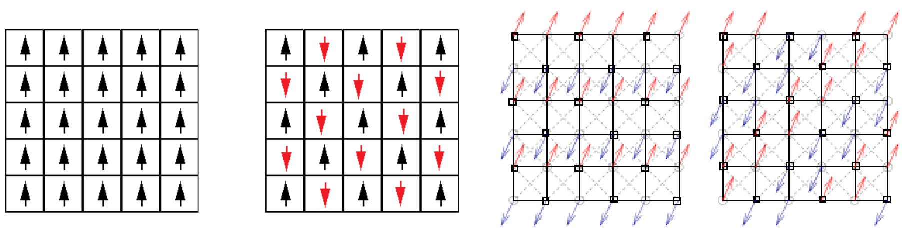
$$\chi \sim |T - T_c|^{-\gamma} , \qquad \gamma = 1.$$



NB: Universality

Table 5.4.1. *Definition of critical exponents and amplitudes.*

Susceptibility	$\chi = \Gamma t^{-\gamma}$	$t > 0$
	$\chi = \Gamma'(-t)^{-\gamma'}$	$t < 0$
Specific heat	$C = \frac{A}{\alpha} t^{-\alpha}$	$t > 0$
	$C = \frac{A'}{\alpha'}(-t)^{-\alpha'}$	$t < 0$
Correlation length	$\xi = \xi_0 t^{-\nu}$	$t > 0$
	$\xi = \xi_0'(-t)^{-\nu'}$	$t < 0$
Order parameter	$\langle\phi\rangle = B(-t)^\beta$	$t < 0$
	$\langle\phi\rangle = D_c^{-1} h\|h\|^{(1-\delta)/\delta}$	$t = 0$
Correlation function	$G(q) = D_\infty q^{-(2-\eta)}$	$t = 0$



$$\gamma = (2 - \eta)\nu$$

$$\gamma + 2\beta = d\nu .$$

Scaling Relations

2D Ising Model:

$\alpha = 0, \ \beta = 1/8, \ \nu = 1, \ \eta = 1/4$

Phase transitions without local order parameters:

1. **Berezinskii-Kosterlitz-Thouless Transition.**

   2D classical XY model

$$H = -J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j = -J \sum_{\langle ij \rangle} \cos(\phi_i - \phi_j)$$

$$\langle S_i \rangle = 0$$ **No long-range order**

$\exists$ finite T_C $\langle S_i S_j \rangle$ → **exponential**

↓

power-law

↓

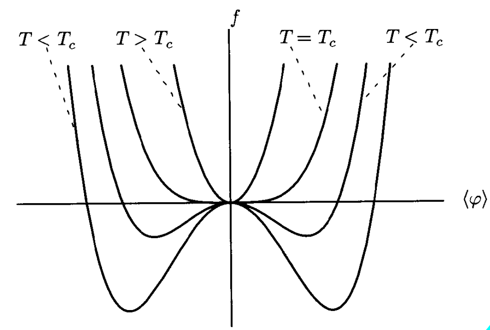
Binding-Unbinding of vortices

↓

Dual (nonlocal) order parameter (Wiegmann, 1977)

Conventional Landau Theory

↓



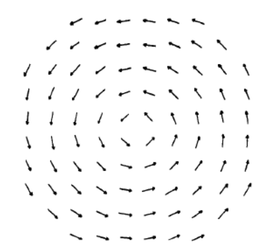


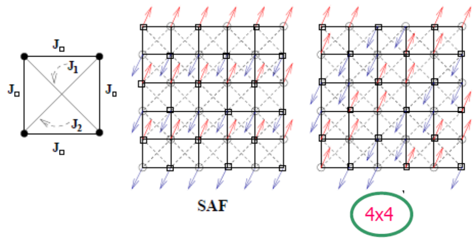
Figure 1. An isolated vortex in the xy model.

J M Kosterlitz and D J Thouless

J. Phys. C: Solid State Phys., Vol. 6, 1973.

2. Frustrated 2D nn+nnn Ising Model

G.Y.C. and C. Gros, Low Temp. Phys. (2005)



FP = Floating Phase: $\langle S(\mathbf{r}'+\mathbf{r})S(\mathbf{r}')\rangle \sim r^{-\eta}\cos(\mathbf{q}\cdot\mathbf{r})$.

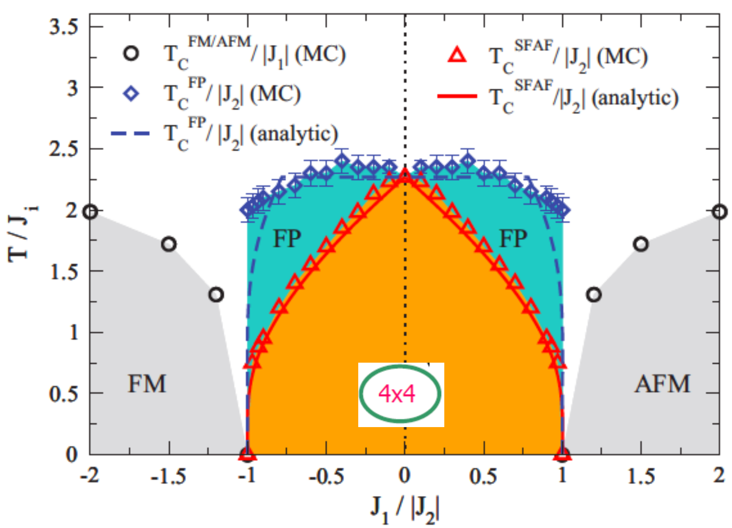
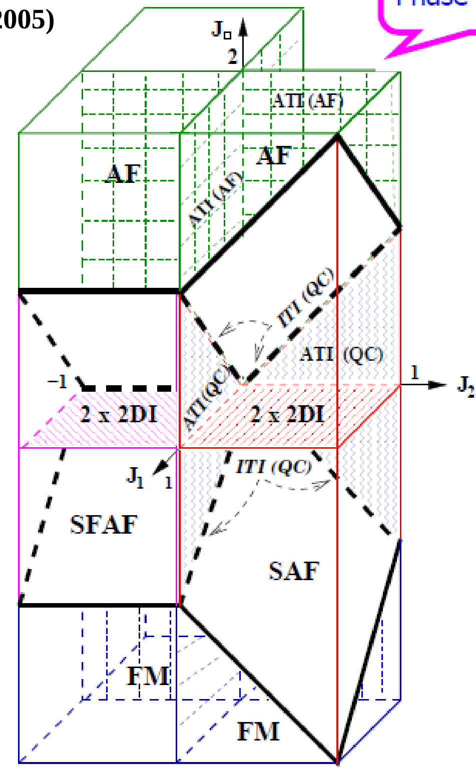


FIG. 2. (Color online) Phase diagram of the anisotropic Ising model for varying NN interactions $J_1/|J_2|$. Three different ground-

A. Kalz and G.Y.C., PRB (2013)

More (analytic) work is warranted (TBA)
Interacting fermions

Ground-State Phase Diagram



PM to FP transition =
BKT transition ->
NO local Order Parameter
Power-Law Correlation function
Analogue: 2D ANNNI model
P. Bak, Rep. Prog. Phys. (1982)

3. Kitaev Model (exactly solvable, free Majorana fermions)

$$H = \sum_{j+l=\text{even}} (J_1 \sigma^x_{j,l}\sigma^x_{j+1,l} + J_2 \sigma^y_{j-1,l}\sigma^y_{j,l} + J_3 \sigma^z_{j,l}\sigma^z_{j,l+1})$$

Kitaev, Ann. Phys, 2003, 2006
Feng, Zhang & Xiang, PRL 2007

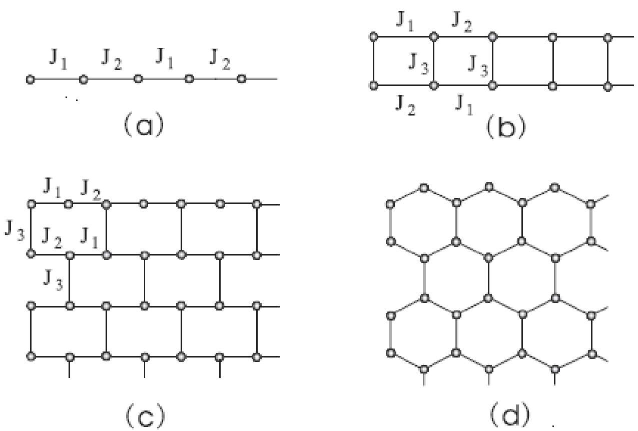


FIG. 1. A brick-wall lattice (c) with its equivalent honeycomb lattice (d). (a) and (b) are the one- and two-row limits of the brick-wall lattices, respectively.

PRL **98,** 087204 (2007) © 2007 The American Physical Society

20 A. Kitaev / Annals of Physics 321 (2006) 2–111

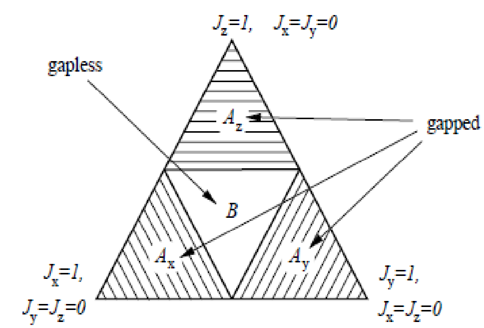
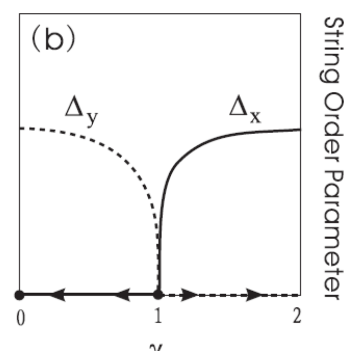


Fig. 5. Phase diagram of the model. The triangle is the section of the positive octant ($J_x, J_y, J_z \geqslant 0$) by the plane $J_x + J_y + J_z = 1$. The diagrams for the other octants are similar.

String order parameters. Local vs nonlocal order

$$\hat{\Delta}_x(j) = \tau^x_0 \tau^x_{2j} = \prod_{k=1}^{2j} \sigma^x_k = (-1)^j \prod_{k=1}^{2j} c_k. \qquad \Delta_x = \lim_{j\to\infty}\langle \hat{\Delta}_x(j)\rangle$$

$$\Delta_y = \lim_{j\to\infty}\left\langle \prod_{k=2}^{2j+1} \sigma^y_k \right\rangle = (-1)^j \lim_{j\to\infty}\left\langle \prod_{k=2}^{2j+1} c_k \right\rangle,$$



$\gamma = J_1 / J_2$

Kitaev Model (SOP)

Jordan Wigner Majorana Fermions

XY spin chain in transverse field
(local Order Parameter)

$\alpha = 0,\ \beta = 1/8,\ \nu = 1,\ \eta = 1/4$

Critical indices of the 2D Ising Model

4. More Examples (you name it):

FQHE

Haldane gapped chain

Mott insulators (1D)

Topological insulators

Heisenberg spin ladders

HUGE (!!!) literature

Extra tools:

Topological numbers

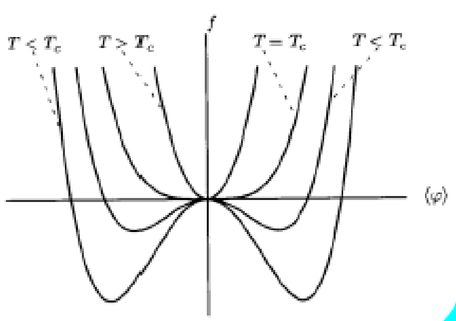
Berry phase

Entropy of entanglement

Entanglement (concurrence)

Summary-I:

Conventional Landau Theory



Noninteracting Kitaev chain:

Kitaev, 2001

$$H = \sum_{n=1}^{N} \Big\{ -\mu(c_n^\dagger c_n - \frac{1}{2}) - t(c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1}) + \Delta(c_{n+1}^\dagger c_n^\dagger + c_n c_{n+1}) \Big\} ,$$

$$H = \frac{i}{2} \sum_{n=1}^{N} \Big\{ \mu a_n b_n - (t+\Delta) b_n a_{n+1} + (t-\Delta) a_n b_{n+1} \Big\}$$

Majorana fermions: $a_n + i b_n \equiv 2 c_n^\dagger$

$$\{a_n, a_m\} = 2\delta_{nm}, \quad \{b_n, b_m\} = 2\delta_{nm},$$
$$\{a_n, b_m\} = 0$$

Jordan-Wigner Transformation:

$$\begin{pmatrix} \sigma_n^x \\ \sigma_n^y \end{pmatrix} = \begin{pmatrix} a_n \\ b_n \end{pmatrix} \prod_{l=1}^{n-1} [i a_l b_l]$$

$$H = -\sum_{i=1}^{N} \left\{ \frac{t}{2} \left[\left(1+\frac{\Delta}{t}\right) \sigma_i^x \sigma_{i+1}^x + \left(1-\frac{\Delta}{t}\right) \sigma_i^y \sigma_{i+1}^y \right] + \frac{1}{2} \mu \sigma_i^z \right\}$$

XY chain in transverse field

Noninteracting Kitaev chain (contd):

DeGottardi, et al (2011)

$$H = \frac{i}{2}\sum_{n=1}^{N}\left\{\mu a_n b_n - (t+\Delta) b_n a_{n+1} + (t-\Delta) a_n b_{n+1}\right\}$$

Majorana string operators: $O_x(m) = \prod_{l=1}^{m-1}[i b_l a_{l+1}]$

$$O_y(m) = \prod_{l=1}^{m-1}[-i a_l b_{l+1}]$$

String Order Parameter (SOP):

$$\mathcal{O}_x^2 = \lim_{(n-m)\to\infty} |\langle O_x(m) O_x(n)\rangle|$$

O_x

Jordan-Wigner Transformation (Spin-Fermion duality):

$$\langle O_x(m) O_x(n)\rangle = \prod_{l=m}^{n-1}[i b_l a_{l+1}] = \langle \sigma_n^x \sigma_m^x\rangle \qquad m_x^2 = \lim_{(m-n)\to\infty} |\langle \sigma_n^x \sigma_m^x\rangle|$$

$$\mathcal{O}_x = \begin{cases} \sqrt{\frac{2}{1+\Delta/t}}\left(\left(\frac{\Delta}{t}\right)^2\left[1-\left(\frac{\mu}{2t}\right)^2\right]\right)^{1/8}; & \Delta/t > 0 \\ 0; & \Delta/t < 0 \end{cases}$$

$\alpha = 0,\ \beta = 1/8,\ \nu = 1,\ \eta = 1/4$

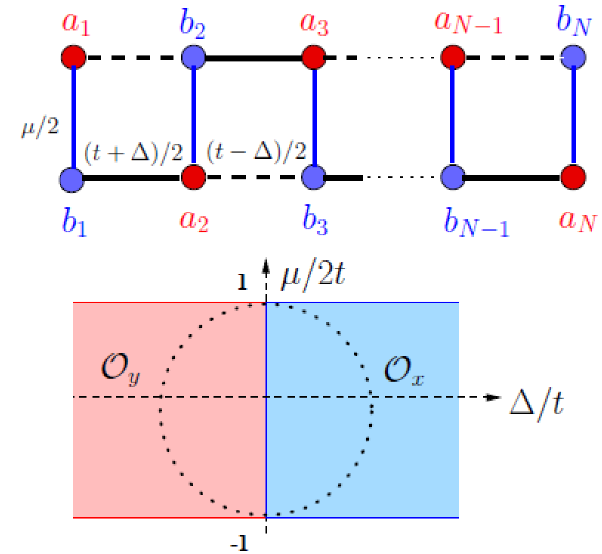
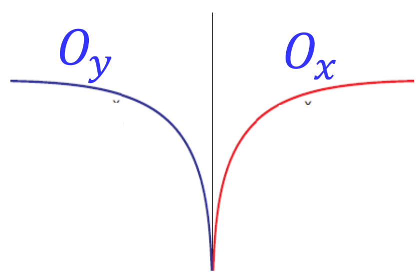


Figure 1: (Color online) The Kitaev chain visualized as a two-leg ladder (upper panel) and its phase diagram (lower panel). The couplings in the ladder are shown according to the Hamiltonian (4). Two phases with nonzero SOPs are shown on the phase diagram along with the disorder line $(\mu/2t)^2 + (\Delta/t)^2 = 1$ (dotted line).

Zero-energy edge Majorana fermions:

$(b_1, a_N)/(a_1, b_N)$



Dimerized XY chain:

$$H = \sum_{i=1}^{N} \frac{J}{4}[(1+\gamma)\sigma_i^x\sigma_{i+1}^x + (1-\gamma)\sigma_i^y\sigma_{i+1}^y + \delta(-1)^i(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y)] + \frac{1}{2}(-1)^i h\sigma_i^z$$

Refs: Perk, et al, Physica (75)
Ye, et al, Comm. Theor. Phys. (02,03)

Jordan-Wigner Transformation:

$$\mathcal{H}(k) = J\cos k\Gamma_3 + J\delta\sin k\Gamma_4 + h\Gamma_5 - J\gamma\sin k\Gamma_{13}$$

$$\epsilon^{\pm}(k) = J\sqrt{\cos^2 k + \left(\sqrt{\left(\frac{h}{J}\right)^2 + \delta^2\sin^2 k} \pm \gamma\sin k\right)^2}$$

Gap:

$$\Delta = J\left|\sqrt{(h/J)^2 + \delta^2} \pm \gamma\right|$$

Lines of Quantum Phase Transitions:

$$\gamma = \pm\sqrt{(h/J)^2 + \delta^2}$$

String Order Parameter:

Refs: den Nijs and Rommelse, PRB (89)
Berg, et al, PRB (08)
GYC and T. Pandey, 2017

String Operator:

$$O_m^\alpha = \exp\left[\frac{i\pi}{2}\sum_{k\leftarrow m}\sigma_k^\alpha\right]$$

String-String Correlation Function:

$$\langle O_m^\alpha O_n^\alpha\rangle = (-1)^{m-1}\left\langle \exp\left[\frac{i\pi}{2}\sum_{k=m}^{n-1}\sigma_k^\alpha\right]\right\rangle$$

String Order Parameter:

$$\mathcal{O}_\alpha^2 = \lim_{l\to\infty}(-1)^l\left\langle \prod_{k=1}^{2l}\sigma_k^\alpha\right\rangle$$

Dimerized XY chain: (*contd)*

$$\sigma_n^x = \tau_{n-1}^x \tau_n^x$$
$$\sigma_n^y = \prod_{l=n}^{N} \tau_l^z \; ,$$

Duality Transformation (E.g., Fradkin & Susskind, PRB(78)):

Dimerized XY Chain (h=0) ⟷ Two Decoupled Ising Chains in Transverse Field (residing on even/odd dual sites)

$$H = H_{\text{even}} + H_{\text{odd}}$$
$$H_{\text{even}} = \frac{1}{4} \sum_{l=1}^{N/2} \left(J^{+-} \tau_{2l-2}^x \tau_{2l}^x + J^{-+} \tau_{2l}^z \right)$$
$$H_{\text{odd}} = \frac{1}{4} \sum_{l=1}^{N/2} \left(J^{++} \tau_{2l-1}^x \tau_{2l+1}^x + J^{--} \tau_{2l-1}^z \right)$$

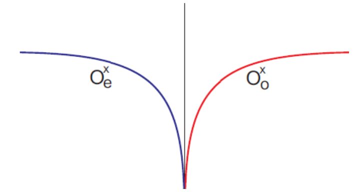
$$J^{\pm\pm} = J(1 \pm \gamma \pm \delta).$$

$Z_2 \otimes Z_2$ symmetry

Nonlocal SOP of XY Chain ⟷ Local (Landau) OP for Ising Chains in Transverse Field

$$\mathcal{O}_x^2 = \lim_{l \to \infty} (-1)^l \langle \tau_0^x \tau_{2l}^x \rangle$$

XY chain: local and nonlocal (String) order parameters



String Order Parameters of the XY chain:

$$\mathcal{O}_{x,e/o} = \begin{cases} 2^{1/4}\left(\frac{t_\mp}{(1+t_\mp)^2}\right)^{1/8} & t_\mp \geq 0 \\ 0 & t_\mp < 0 \end{cases}$$

$$\mathcal{O}_{y,e/o} = \begin{cases} 0 & t_\mp \geq 0 \\ \mathcal{O}_{x,e/o}(-t_\mp) & t_\mp < 0 \end{cases}$$

$$t_\pm = \gamma \pm \delta$$

Magnetization:

$$m_\alpha = \mathcal{O}_{\alpha,o}\mathcal{O}_{\alpha,e}\ , \quad \alpha = x, y.$$

Topological winding number:

$$N_w = N_- + N_+ = \frac{1}{2}[\text{sign}(\delta - \gamma) + \text{sign}(\delta + \gamma)]$$

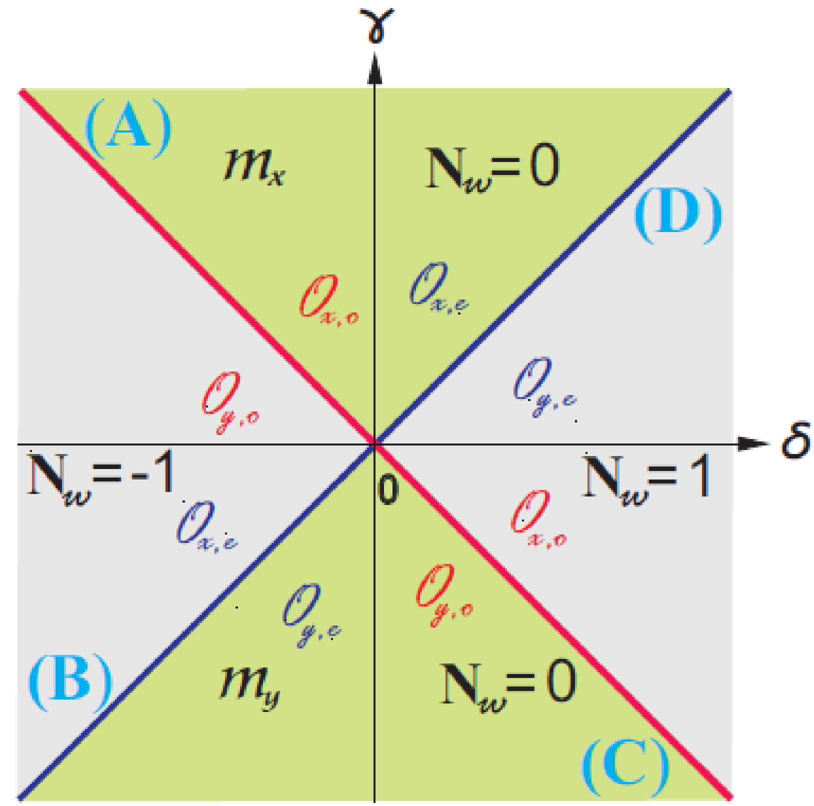


FIG. 1: (Color online) Phase diagram of the anisotropic dimerized XY chain. Nonvanishing string and local order parameters, topological winding numbers are shown in four sectors A,B,C,D of the (δ, γ) parametric plane. The violet/red lines $\gamma = \pm\delta$ are the lines of quantum phase transitions (gaplessness).

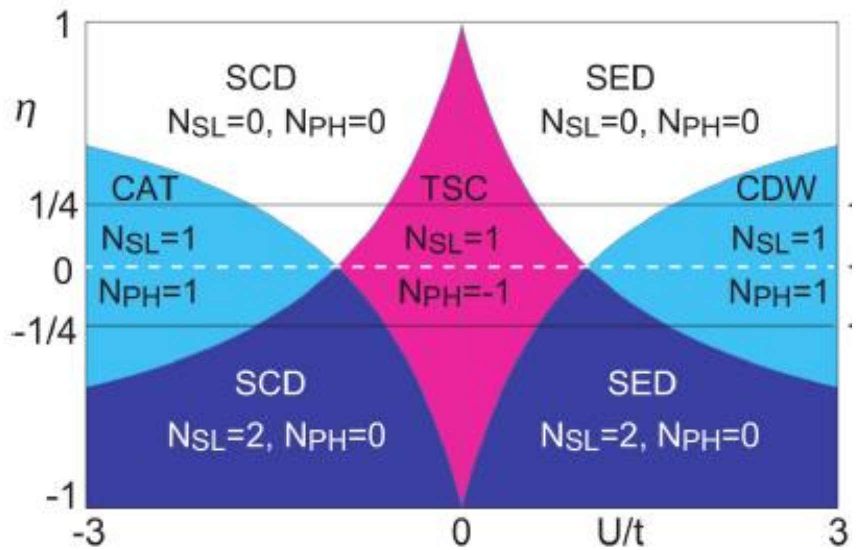
Dimerized interacting Kitaev chain:

Miao, et al , PRL (2017); 1707.08430
Ezawa, 1707.03983

$$H = \sum_{n=1}^{N} \Big\{ -\mu\big(c_n^\dagger c_n - \frac{1}{2}\big) - t_n\big(c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1}\big) + \Delta_n\big(c_{n+1}^\dagger c_n^\dagger + c_n c_{n+1}\big) + U_n\big(2c_{n+1}^\dagger c_{n+1} - 1\big)\big(2c_n^\dagger c_n - 1\big)\Big\}, \quad (16)$$

where

$$t_n = t[1-(-1)^n\delta], \quad \Delta_n = \Delta[1-(-1)^n\delta], \qquad U_n = U[1-(-1)^n\delta]. \quad (17)$$



Ezawa, 1707.03983

Special symmetric point: $\Delta = t, \ \mu = 0.$

$$H = \sum_{n=1}^{N} \Big[-t_n\sigma_n^x\sigma_{n+1}^x + U_n\sigma_n^y\sigma_{n+1}^y \Big]$$

Dimerized interacting Kitaev chain (contd):

Special symmetric point: $\Delta = t,\ \mu = 0.$

$$H = \sum_{n=1}^{N} \left[-t_n i b_n a_{n+1} + U_n i a_n b_n i a_{n+1} b_{n+1} \right] = \sum_{n=1}^{N} \left[-t_n \sigma_n^x \sigma_{n+1}^x + U_n \sigma_n^z \sigma_{n+1}^z \right],$$

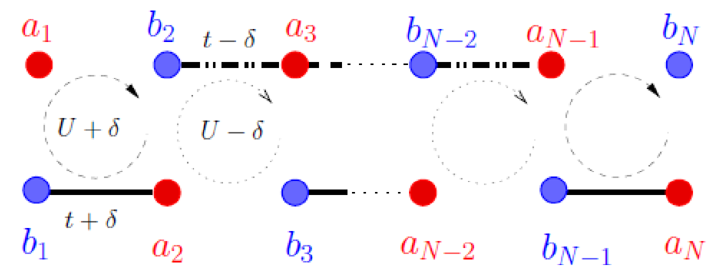


Figure 2: (Color online) The interacting dimerized Kitaev chain visualized as a two-leg ladder. The in-leg and plaquette couplings in the ladder are shown according to the Hamiltonian (19).

$$H = \sum_{n=1}^{N} \left[-t_n \sigma_n^x \sigma_{n+1}^x + U_n \sigma_n^y \sigma_{n+1}^y \right]$$

Majorana string operators:

$$O_{x,e}(m) = \prod_{l=1}^{m} [i b_{2l-1} a_{2l}]$$

$$O_{x,o}(m) = \prod_{l=1}^{m} [i b_{2l} a_{2l+1}]$$

Duality transformation:

$$\sigma_n^x = \tau_{n-1}^x \tau_n^x$$

$$\sigma_n^y = \prod_{l=n}^{N} \tau_l^z ,$$

$$H = H_e + H_o$$

$$H_e = \sum_{l=1}^{N/2} -t(1+\delta)\tau_{2l-2}^x \tau_{2l}^x + U(1-\delta)\tau_{2l}^z$$

$$H_o = \sum_{l=1}^{N/2} -t(1-\delta)\tau_{2l-1}^x \tau_{2l+1}^x + U(1+\delta)\tau_{2l-1}^z$$

$Z_2 \otimes Z_2$ symmetry

$$\varepsilon_{e/o}(k) = \frac{1}{2} t(1 \pm \delta) \sqrt{\sin^2 k + \left(\cos k - \frac{U}{t} \frac{1 \mp \delta}{1 \pm \delta} \right)^2}$$

Dimerized interacting Kitaev chain (contd):

SOPs and local OPs:

$$\mathcal{O}^2_{x,e/o} = \lim_{(R-L)\to\infty} \langle \tau^x_L \tau^x_R \rangle$$

$$L = 2n,\ R = 2m \ \longmapsto\ \mathcal{O}_{\alpha,e}$$

$$L = 2n-1,\ R = 2m-1 \ \longmapsto\ \mathcal{O}_{\alpha,o}$$

$$\langle \tau^y_L \tau^y_R \rangle = \langle \prod_{l=L+1}^{R} [-i a_l b_l] \rangle$$

$$\xrightarrow{(R-L)\to\infty} [-\mathrm{sign}(U)]^{\frac{R-L}{2}} \mathcal{O}^2_{z,e/o} \ ,$$

$$\mathcal{O}_{x,e/o} = \left(1 - \left[\frac{U}{t}\frac{1\mp\delta}{1\pm\delta}\right]^2\right)^{1/8} ,$$

at $\{\delta \gtrless \delta_{c,o}\} \cup \{\delta \gtrless \delta_{c,e}\}$

$$\mathcal{O}_{z,e/o} = \left(1 - \left[\frac{t}{U}\frac{1\pm\delta}{1\mp\delta}\right]^2\right)^{1/8} ,$$

at $\{\delta \lessgtr \delta_{c,o}\} \cup \{\delta \lessgtr \delta_{c,e}\}$

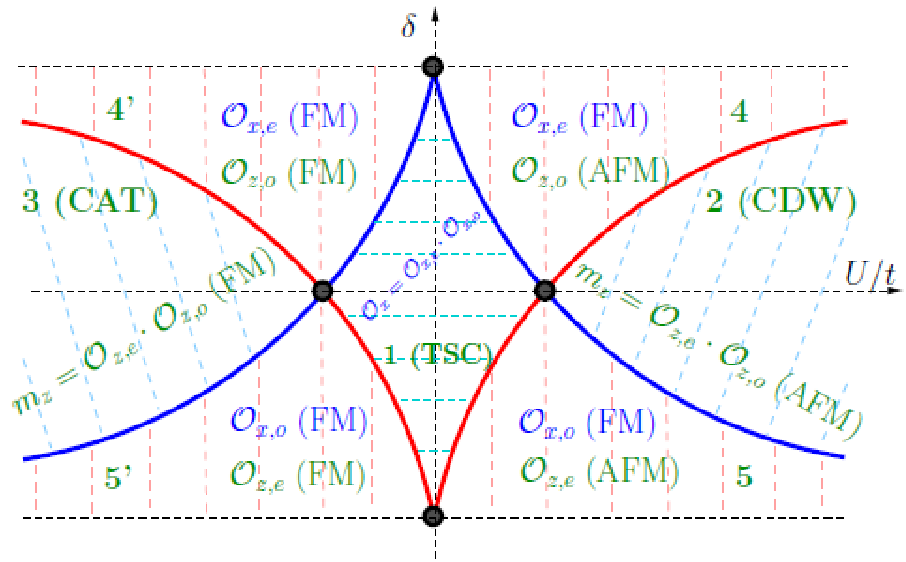


Figure 3: (Color online) Phase diagram of the interacting dimerized Kitaev model with nonvanishing order parameter shown for each phase.

$$\mathcal{O}_x = \left[\left(1 - \left[\frac{U}{t}\frac{1-\delta}{1+\delta}\right]^2\right)\left(1 - \left[\frac{U}{t}\frac{1+\delta}{1-\delta}\right]^2\right)\right]^{1/8}$$

at $\{\delta_{c,e} < \delta < \delta_{c,o}\} \cup \{|U|/t < 1\}$.

$$m_z = |\langle 2c^\dagger_n c_n - 1\rangle| = |\langle a_n b_n\rangle| = |\langle \tau^y_e\rangle\langle\tau^y_o\rangle|$$

$$= \left[\left(1 - \left[\frac{t}{U}\frac{1+\delta}{1-\delta}\right]^2\right)\left(1 - \left[\frac{t}{U}\frac{1-\delta}{1+\delta}\right]^2\right)\right]^{1/8}$$

at $\{\delta_{c,o} < \delta < \delta_{c,e}\} \cup \{|U|/t > 1\}$.

Dimerized Two-Leg Ladders:

Refs: Martin-Delgado, et al (96-08)
GYC and T. Pandey, 2017

$$H = \sum_{n=1}^{N}\sum_{\alpha=1}^{2} J_\alpha(n)\mathbf{S}_\alpha(n)\cdot\mathbf{S}_\alpha(n+1) + J_\perp \sum_{n=1}^{N}\mathbf{S}_\alpha(n)\cdot\mathbf{S}_{\alpha+1}(n),$$

$$J_\alpha(n) = J[1-(-1)^{n+\alpha}\delta] \quad \text{(staggered)}$$
$$J_\alpha(n) = J[1-(-1)^{n}\delta] \quad \text{(columnar)}$$

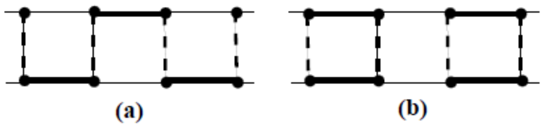


FIG. 2: Dimerized two-leg ladder. Bold/thin/dashed lines represent the stronger/weaker chain coupling $J(1\pm\delta)$ and rung coupling $J_\perp$, respectively. Dimerization patterns: (a) - staggered; (b)- columnar.

Jordan-Wigner Transformation + Mean-Field Approximation:

Staggered Ladder:

$$\mathcal{H}^{st}(k) = \frac{1}{2}J_{\perp R}\Gamma_1 + J_R\cos k\Gamma_3 + J_R\delta\sin k\Gamma_{35}$$

$$\epsilon^\pm(k) = J_R\sqrt{\cos^2 k + \left(\delta\sin k \pm \frac{J_{\perp R}}{2J_R}\right)^2}$$

Gap: $$\Delta = J_R\left|\delta \pm \frac{J_{\perp R}}{2J_R}\right|$$

Lines of Quantum Phase Transitions:

$$\frac{J_{\perp R}}{2J_R} = \pm\delta$$

See GYC, et al, PRB (08,11)

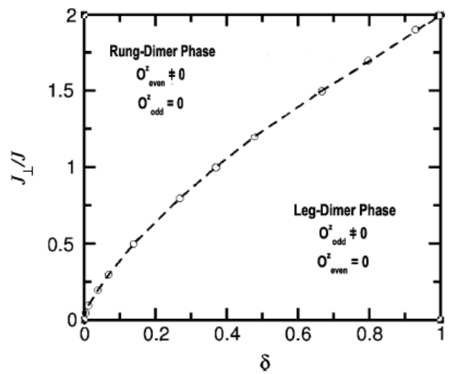


FIG. 4. Two-leg ladder; critical line $J_{\perp c}(\delta)$ where the gap of the staggered phase vanishes. Adapted from Ref. 12, original data from Ref. 9.

Columnar Ladder:

$$\mathcal{H}^{col}(k) = \frac{1}{2}J_{\perp R}\Gamma_1 + J_R\cos k\Gamma_3 + J_R\delta\sin k\Gamma_4$$

$$\epsilon(k) = J_R\sqrt{\cos^2 k + \delta^2\sin^2 k + \left(\frac{J_{\perp R}}{2J_R}\right)^2}$$

Gap: $$\Delta = J_R\sqrt{\delta^2 + \left(\frac{J_{\perp R}}{2J_R}\right)^2}$$

Lines of Quantum Phase Transitions-**None**
(Always Gapped)

String Order Parameters (Staggered Ladders):

Effective Free-Fermionic Hamiltonian (mean-field) in the Majorana representation :

$$
\begin{aligned}
H^{\rm st} &= \frac{i}{4} J_R(1-\delta) \sum_{l=1}^{N} (a_{2l-1} b_{2l+2} + a_{2l+2} b_{2l-1}) \\
&- \frac{i}{4} J_R(1+\delta) \sum_{l=1}^{N} (a_{2l} b_{2l+1} + a_{2l+1} b_{2l}) \\
&- \frac{i}{4} J_{\perp R} \sum_{l=1}^{N} (a_{2l-1} b_{2l} + a_{2l} b_{2l-1}) \ ,
\end{aligned}
$$

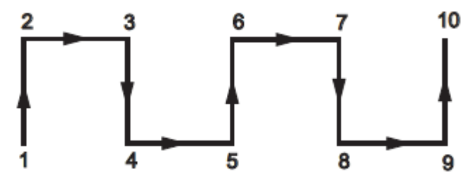


FIG. 3: Path of the Jordan-Wigner transformation used for the two-leg ladder.

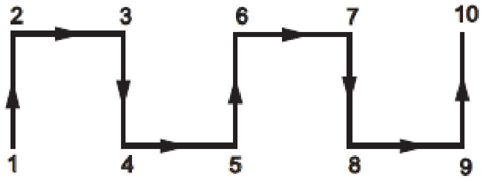
Inverse Jordan-Wigner transformation from Majorana to Dual Spins:

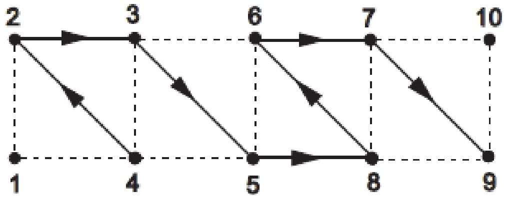
$$
\begin{aligned}
\sigma^x_n \sigma^x_{n+1} &= i b_n a_{n+1} = \tau^x_{n-1} \tau^x_{n+1} \\
\sigma^y_n \sigma^y_{n+1} &= -i a_n b_{n+1} = \tau^z_n \ .
\end{aligned}
$$

$$
\begin{aligned}
H^{\rm st} &= H^{\rm st}_{\rm e} + H^{\rm st}_{\rm o} \\
H^{\rm st}_{\rm e} &= \frac{1}{4} \sum_{l=1}^{N} (J_{\perp R} \tau^x_{2l-2} \tau^x_{2l} + J_R(1-\delta) \tau^x_{2l-2} \tau^z_{2l} \tau^x_{2l+2} + J_R(1+\delta) \tau^z_{2l}) \\
H^{\rm st}_{\rm o} &= \frac{1}{4} \sum_{l=1}^{N} (J_R(1+\delta) \tau^x_{2l-1} \tau^x_{2l+1} - J_R(1-\delta) \tau^y_{2l-1} \tau^y_{2l+1} + J_{\perp R} \tau^z_{2l-1}) \ .
\end{aligned}
$$

$Z_2 \otimes Z_2$ symmetry

Two different paths used for String Operators:





String Order Parameters (mutually exclusive):

$$\mathcal{O}^2_{x,e} = \lim_{N\to\infty} (-1)^N \left\{ \left\langle \prod_{k=1}^{2N} \sigma_k^x \right\rangle = \langle \tau_0^x \tau_{2N}^x \rangle \right\}$$

$$\mathcal{O}^2_{x,o} = \lim_{N\to\infty} (-1)^{N-1} \left\{ \left\langle \prod_{k=2}^{2N-1} \sigma_k^x \right\rangle = \langle \tau_1^x \tau_{2N-1}^x \rangle \right\}$$

$$\mathcal{O}_{x,e} = \begin{cases} \left[\frac{\kappa^2-\delta^2}{1+\kappa^2-\delta^2}\right]^{1/8} & |\kappa/\delta| \geq 1 \\ 0 & |\kappa/\delta| < 1 \end{cases}$$

$$\mathcal{O}_{x,o} = \begin{cases} 0 & |\kappa/\delta| > 1 \\ \sqrt{\frac{2}{1+\delta}}\,[\delta^2 - \kappa^2]^{1/8} & |\kappa/\delta| \leq 1 \end{cases}$$

Critical indices of the 2D Ising Model

Topological winding number:

$$N_w = N_- + N_+ = \frac{1}{2}\left[\text{sign}(\delta - \kappa) + \text{sign}(\delta + \kappa)\right]$$

$$\kappa \equiv \frac{J_{\perp R}}{2J_R}$$

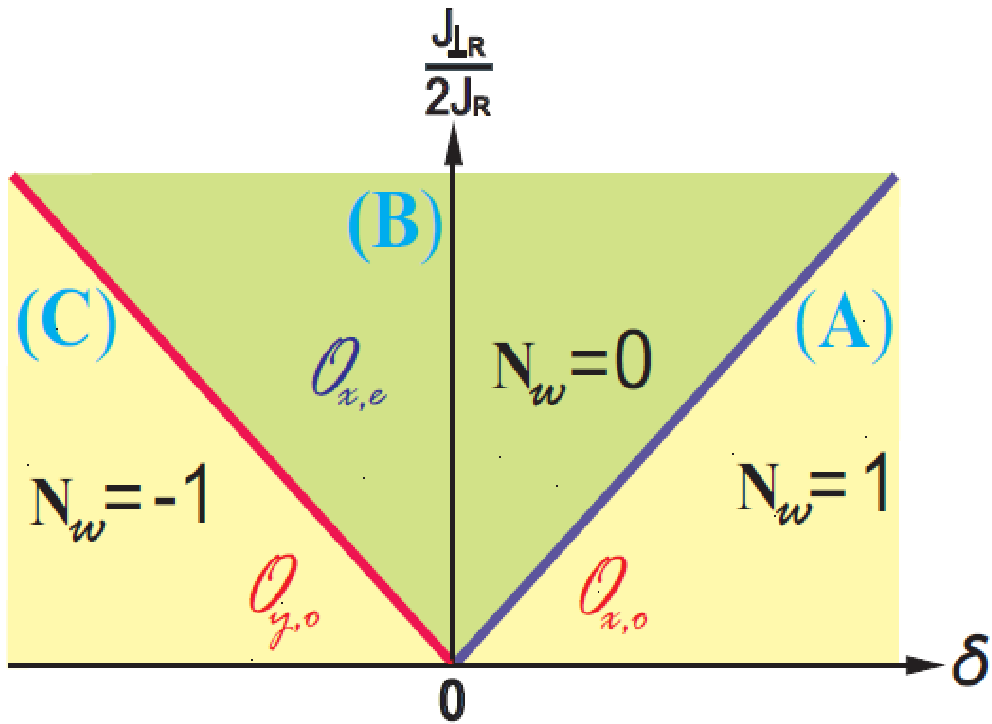


FIG. 4: (Color online) Phase diagram of the two-leg staggered ladder in the parametric plane $(J_{\perp R}/2J_R, \delta)$. The violet/red lines $J_{\perp R}/2J_R = \pm\delta$ are the lines of quantum phase transitions (gaplessness) separating two phases (B) and (A,C) characterized by their distinct SOPs and winding numbers.

Model Hamiltonians: Similarities & Extensions

Dimerized XY chain

$$\mathcal{H}(k) = J\cos k\Gamma_3 + J\delta\sin k\Gamma_4 + h\Gamma_5 - J\gamma\sin k\Gamma_{13}$$

↕

Bogoliubov-de Gennes Hamiltonian of topological SC (Class DIII)

$$\mathcal{H}(k) = \begin{pmatrix} \hat{\mathfrak{h}}(k) & \hat{\Delta}(k) \\ \hat{\Delta}^\dagger(k) & -\hat{\mathfrak{h}}(k) \end{pmatrix}$$

$$\hat{\mathfrak{h}}(k) \equiv J\cos k\sigma_1 + J\delta\sin k\sigma_2 + h\sigma_3 \ ,$$

$$\hat{\Delta}(k) \equiv -iJ\gamma\sin k\sigma_1 \ .$$

Staggered 2-leg ladder (approximate)

$$\mathcal{H}^{\rm st}(k) = \frac{1}{2}J_{\perp R}\Gamma_1 + J_R\cos k\Gamma_3 + J_R\delta\sin k\Gamma_{35}$$

↕

TI/SC Hamiltonians, Kitaev ladder

$$\tilde{\mathcal{H}}^{\rm st}(k) = \begin{pmatrix} \hat{\mathfrak{h}}_+(k) & 0 \\ 0 & \hat{\mathfrak{h}}_-(k) \end{pmatrix}$$

$$\hat{\mathfrak{h}}_\pm(k) = J_R\cos k\sigma_1 - (J_R\delta\sin k \pm \frac{1}{2}J_{\perp R})\sigma_2$$

String Order Parameters Calculations:

$Z_2 \otimes Z_2$ symmetry

2-leg ladder

$$\begin{aligned} H^{\rm st} &= \frac{i}{4} J_R(1-\delta) \sum_{l=1}^{N} (a_{2l-1} b_{2l+2} + a_{2l+2} b_{2l-1}) \\ &- \frac{i}{4} J_R(1+\delta) \sum_{l=1}^{N} (a_{2l} b_{2l+1} + a_{2l+1} b_{2l}) \\ &- \frac{i}{4} J_{\perp R} \sum_{l=1}^{N} (a_{2l-1} b_{2l} + a_{2l} b_{2l-1}) \ , \end{aligned}$$

$$\begin{aligned} H^{\rm st} &= H^{\rm st}_{\rm e} + H^{\rm st}_{\rm o} \\ H^{\rm st}_{\rm e} &= \frac{1}{4} \sum_{l=1}^{N} \left(J_{\perp R} \tau^x_{2l-2} \tau^x_{2l} + J_R(1-\delta) \tau^x_{2l-2} \tau^z_{2l} \tau^x_{2l+2} + J_R(1+\delta) \tau^z_{2l} \right) \\ H^{\rm st}_{\rm o} &= \frac{1}{4} \sum_{l=1}^{N} \left(J_R(1+\delta) \tau^x_{2l-1} \tau^x_{2l+1} - J_R(1-\delta) \tau^y_{2l-1} \tau^y_{2l+1} + J_{\perp R} \tau^z_{2l-1} \right) \ . \end{aligned}$$

Transverse Ising chain with three-spin interaction

$$H = \sum_{i=1}^{N} \left(J \sigma^x_i \sigma^x_{i+1} + J_3 \sigma^x_{i-1} \sigma^z_i \sigma^x_{i+1} + h \sigma^z_i \right)$$

$$m_x^2 = \lim_{N \to \infty} \langle \sigma^x_1 \sigma^x_{N+1} \rangle$$

the $N \times N$ Toeplitz determinant:[41]

$$\langle \sigma^x_1 \sigma^x_{N+1} \rangle = \begin{vmatrix} G_{-1} & \dots & G_{-N} \\ \vdots & \cdots & \vdots \\ G_{N-2} & \dots & G_{-1} \end{vmatrix}$$

$$G_{l-m} \equiv i \langle b_l a_m \rangle$$

$$m_x^2 = \left[\frac{J^2 - (J_3 - h)^2}{J^2 + 4hJ_3} \right]^{1/4}$$

Generalization:

n-leg ladders/tubes → Ising chains with multi-spin interactions (work in progress)

Conclusions & Future work:

1. **Dimerized Kitaev and XY chain:**
   Exact SOPs and Landau (local) OPs calculation.
   Winding numbers.

2. **Dimerized Ladders: Mean-Field approximation ->**
   Similar Program. Analytical results for SOPs (!!)

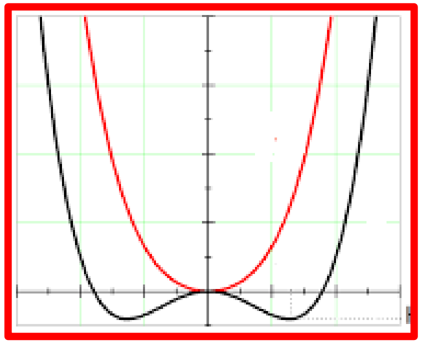
3. **Unifying framework: local and nonlocal OPs, hidden symmetries.**

$$H = \sum_{n=1}^{N}\sum_{\alpha=1}^{m} J_\alpha(n)\mathbf{S}_\alpha(n)\cdot\mathbf{S}_\alpha(n+1) + J_\perp \sum_{n=1}^{N}\sum_{\alpha=1}^{m-1}\mathbf{S}_\alpha(n)\cdot\mathbf{S}_{\alpha+1}(n),$$

$$H = \sum_{n=1}^{N}\Big\{ -\mu\big(c_n^\dagger c_n - \tfrac{1}{2}\big) - t_n\big(c_{n+1}^\dagger c_n + c_n^\dagger c_{n+1}\big) + \Delta_n\big(c_{n+1}^\dagger c_n^\dagger + c_n c_{n+1}\big) + U_n\big(2c_{n+1}^\dagger c_{n+1} - 1\big)\big(2c_n^\dagger c_n - 1\big)\Big\},$$

$$\begin{aligned} H^{\rm st} = & \frac{i}{4}J_R(1-\delta)\sum_{l=1}^{N}(a_{2l-1}b_{2l+2} + a_{2l+2}b_{2l-1}) \\ & - \frac{i}{4}J_R(1+\delta)\sum_{l=1}^{N}(a_{2l}b_{2l+1} + a_{2l+1}b_{2l}) \\ & - \frac{i}{4}J_{\perp R}\sum_{l=1}^{N}(a_{2l-1}b_{2l} + a_{2l}b_{2l-1}) , \end{aligned}$$

$$\begin{aligned} H &= H_{\rm even} + H_{\rm odd} \\ H_{\rm even} &= \frac{1}{4}\sum_{l=1}^{N/2}\big(J^{+-}\tau^x_{2l-2}\tau^x_{2l} + J^{-+}\tau^z_{2l}\big) \\ H_{\rm odd} &= \frac{1}{4}\sum_{l=1}^{N/2}\big(J^{++}\tau^x_{2l-1}\tau^x_{2l+1} + J^{--}\tau^z_{2l-1}\big) \end{aligned}$$



4. **Similar Hamiltonians: chains, ladders, topological insulators/**
   superconductors.

5. **Work in progress: spin tubes and n-leg ladders.**

THE END

THANK YOU!