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A Weak Order Approach to Group Ranking

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Abstract—A new approach to the problem of *a group ranking* is presented. The solution satisfies four of Arrow's six axioms [1,2] and is sufficient for a wide range of applications. A constructive algorithm is proposed for finding a group ranking on the basis of individual rankings of which some may collide with others.

Keywords—Group ranking, Arrow's impossibility theorem, Arrow's axioms, Social function, Partial and weak orders, Algorithm.

1. INTRODUCTION

The basic model of knowledge engineering is based on teamwork in which a knowledge engineer mediates between human experts and the knowledge base. The knowledge engineer elicits knowledge from the experts, refines it with them, and represents it in the knowledge base. Oddly enough, the problem of a group ranking has not been explored deeply enough. Arrow's impossibility theorem (see [1,2]) proves that no solution exists under general assumptions (see Section 4). However, a constructive algorithm exists under modified (but still practical) assumptions. One such algorithm is presented in this paper, which is an extended and refined version of [3].

Let us assume that there is a finite set X of objects (e.g., criteria or stimuli) and m experts, numbered $1, 2, \ldots, m$. Each expert i compares objects and constructs a *nonempty weak* order $<_i \subseteq X \times X$. The problem of a group ranking is to define one *group ranking*, $<_g \subseteq X \times X$, which is a "compromise" based on all of the individual rankings $\mathrm{IR} = \{<_1, \ldots, <_m\}$. It is possible that some of the individual rankings may collide with other individual rankings. The group ranking $<_g$ is expected to be a *weak order*, but it should be somehow consistent with most (if not all) of the individual rankings IR.

Various instances of the above problem occur in the Social Choice Theory, Decision Making Process and Knowledge-Based Systems (see [1,2,4,5] and many others). Also a similar problem, but with quite different consistency rules, occurs in the Theory of Concurrency, where $<_i$ rep-

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resents individual equivalent observations and $<_g$ is a 'concurrent history' (see [6]). Compliance with the consistency rules is crucial since the Arrow's Impossibility Theorem [1,2] says that under certain assumptions (called Arrow's Axioms; see, for example, [2]) the above formulated problem has no general solution. Needless to say, a constructive solution should be of interest to both theoreticians and practitioners.

Approximation of discrete relations by preference orders reduces disorder and randomness in judgements. This is useful for any inference tool applicable to knowledge-based systems. It is easier to implement and use partial orders which may be programmed (in most instances) as a block of *if-the-else*, *switch*, or *case* constructs (in some programming languages).

In this paper, we present a range of fairly general solutions satisfying four of Arrow's six axioms. They are, in our opinion, sufficient for a wide range of applications. In particular, the solutions presented may be successfully applied to knowledge-based systems and used in the decision making process.

2. RANKING RELATIONS AND PREFERENCE ORDERS

A relation $< \subseteq X \times X$ is called a *(sharp) partial order* if it is irreflexive and transitive, i.e., if $a < b \Rightarrow \neg b < a$, and $a < b < c \Rightarrow a < c$, for all $a, b, c \in X$. We will write $a \sim b$ if $\neg a < b \wedge \neg b < a \wedge a \neq b$ (where $\neg$ denotes *not*), that is if a and b are distinct incomparable elements of X. A partial order is:

- *total* if $\sim$ is empty, i.e., for all $a, b \in X$. $a < b \vee b < a$,
- *weak* [7] if $a \sim b \sim c \Rightarrow a \sim c \vee a = c$, i.e., if $\sim \cup\, \mathrm{id}_X$ is an equivalence relation (id_X is an identity on X).

Evidently, every total order is weak. A wide and complete analysis of axioms for *preference* and *indifference* relations is presented in [2]. It is worthwhile noting that in the majority of cases the preference relation is just a weak order, while indifference corresponds to $\sim \cup\, \mathrm{id}_X$. In some cases (such as concurrency theory, see [6]), it is more realistic to model preferences by semiorders or interval orders [7]. However, here we assume that preference is modeled by weak orders. The charm of weak orders is that they can be represented uniquely by *value functions* (compare, for example, [2]). More formally, a partial order $< \subseteq X \times X$ is weak if and only if there is a total order $<_t \subseteq Y \times Y$, and a mapping (value function) $v : X \to Y$, such that $a < b \Leftrightarrow v(a) <_t v(b)$. Weak orders can easily be represented by *step-sequences*. For instance, if $X = \{a, b, c, d, e\}$, and $v(a) = v(c) = 0$, $v(e) = 2$, $v(b) = v(d) = 5$, then the weak order $<$ is uniquely represented by the step-sequence $\{a, c\}\{e\}\{b, d\}$ (see [6]).

The first step in our approach is to define and construct a *ranking relation*.

Let X be our set of objects to be ranked and $\mathrm{IR} = \{<_1, \dots, <_m\}$ be the set of individual (weakly ordered) rankings. By a (*binary*) *ranking relation* over the set IR, we mean any relation $R \subseteq X \times X$ satisfying the following constraints:

- $\forall x, y \in X.\ |\{i \mid x <_i y\}| \geq |\{i \mid y <_i x\}| \Rightarrow \neg yRx$, and
- $\forall x, y \in X.\ (\forall i = 1, \dots, m.\ x <_i y) \Rightarrow xRy$,

where for every set A, $|A|$ denotes the number of elements of A.

An obvious example of a ranking relation is R_{smv}, *simple majority voting relation*, where

$$xR_{\mathrm{smv}}y \Leftrightarrow |\{i \mid x <_i y\}| > |\{i \mid y <_i x\}|,$$

i.e., $xR_{\mathrm{smv}}y$ if more experts preferred y over x. The relation R_{smv} exists for every IR, and every ranking relation over IR satisfies $R \subseteq R_{\mathrm{smv}}$. R_{smv} is not, however, the only ranking relation one may think of. If, for instance, 10 experts would say x is better than y, and 11 just the opposite, the ranking that x and y are of the same value seems to be the most appropriate in many cases (except for sports and politics!).

In general, one would consider the value of $|\{i \mid x <_i y\}| - |\{i \mid y <_i x\}|$ to establish the relation between x and y. If it is "big," we assume xRy. For "small" or "very small," we would rather assume that $\neg xRy$ and $\neg yRx$. Note that in both cases $\neg yRx$.

The second condition says that if all experts prefer y over x, then the entire group does, so xRy. The relation R is always irreflexive, i.e., $\neg xRx$.

If, by chance, the relation R is a weak order, our goal has been achieved. Unfortunately, the relation R may not even be a partial order. The precise definition of R depends on the specific properties of the set X.

Unanimity has a central place in ranking theory and practice. If every expert prefers y to x, then the group should prefer y to x (see *Pareto's principle* [2]). Let us define the relation $<_U \subseteq X \times X$, as

$$<_U = <_1 \cap \cdots \cap <_m .$$

The relation $<_U$ will be called *unanimous preference*. Unanimous preference is always a partial order (since all $<_i$ are partial orders), although it is not necessarily a weak order. The following corollary follows immediately from the definition of $<_U$ and the second constraint on R.

COROLLARY 2.1.

(1) $\forall x, y \in X.\ (x <_1 y \wedge \cdots \wedge x <_m y) \Leftrightarrow x <_U y$.
(2) $<_U \subseteq R$. ∎

We will require that the group ranking $<_g$ satisfies $<_U \subseteq <_g$.

3. APPROXIMATION OF RANKING RELATIONS BY PREFERENCE ORDERS

As we mentioned above, the ranking relation R may not be a partial order. Our goal is to find a relation $<_g$ which is not only a weak order, but also "the best" approximation of R. This will be done in two steps. First, we will find $<^U_{R^+}$, "the best" partial order approximation of R. Next, we will discuss various weak order approximations of $<^U_{R^+}$.

The problem is that the set X is believed to be partially ordered, but the data acquisition process is so influenced by informational noise, imprecision, randomness, or expert ignorance, that the collected data R is only some relation on X. We may say that R gives a fuzzy picture, and to focus it, we must do some pruning and/or extending. If R is not a partial order, then either it is not reflexive, or it is not transitive, or both. Suppose that R is not transitive. This could be seen as a limitation of the discriminatory power of the data acquisition process. Let us define the relation $R^+ = \bigcup_{i=1}^{\infty} R^i$, where $R^{i+1} = R^i \circ R$, and $\circ$ denotes composition of relations. Evidently $R \subseteq R^+$ and R^+ is transitive. If R is transitive, then $R^+ = R$. The relation R^+ may not be irreflexive. If R^+ is not irreflexive, we have $\exists x, y \in X.\ xR^+y \wedge yR^+x$. Such a situation could be interpreted as a side effect of the data noise, uncertainty, randomness, etc.

PROPOSITION 3.1. [8]. *Let $Q \subseteq X \times X$ be a transitive relation. Define: $x <_Q y \Leftrightarrow xQy \wedge \neg yQx$. The relation $<_Q$ is a partial order (i.e., it is irreflexive and transitive).* ∎

The relation $<_{R^+}$, defined as $x <_{R^+} y \Leftrightarrow xR^+y \wedge \neg yR^+x$, will be called a *partial order approximation of* (*ranking relation*) R. We have $<_{R^+} \subseteq R^+$, and if R^+ is irreflexive, then $R^+ = <_{R^+}$.

COROLLARY 3.2. $<_U \subseteq R \subseteq R^+$. ∎

However, if R^+ is not irreflexive, it may happen that $<_U \setminus <_{R^+} \neq \emptyset$, so $<_U$ may be not included in $<_{R^+}$.

Let us define $<^U_{R^+}$ as the *smallest partial order* satisfying the following constraint:

$$<_U \cup <_{R^+} \subseteq <^U_{R^+} .$$

PROPOSITION 3.3. $(<_U \cup <_{R^+})^+ = <^U_{R^+}$.

PROOF. $(<_U \cup <_{R^+})^+$ is evidently the smallest transitive relation containing $<_U \cup <_{R^+}$. It suffices to show that $(<_U \cup <_{R^+})^+$ is irreflexive. Suppose it is not, i.e., $\exists x, y \in X$ such that $x(<_U \cup <_{R^+})^+ y \wedge y(<_U \cup <_{R^+})^+ x$. This means there are $n > 0$, $0 \leq k \leq n$, $x_0, \ldots, x_n \in X$, $x_0 = x$, $x_k = y$, and $Q_1, \ldots, Q_n \subseteq X \times X$, such that

$$x_0 Q_1 x_1 Q_2 x_2 \ldots x_{k-1} Q_k x_k \ldots x_{n-1} Q_n x_0,$$

where Q_i is either $<_U$ or $<_{R^+}$. Both $<_U$ and $<_{R^+}$ are transitive; therefore, at least one Q_i, say Q_{i_0}, must be equal to $<_{R^+}$, i.e., $Q_{i_0} = <_{R^+}$. Because $<_U \subseteq R^+$ and $<_{R^+} \subseteq R^+$, all $Q_i \subseteq R^+$, so we have $\forall i, j \leq n.\ x_i R^+ x_j \wedge x_j R^+ x_i$. In particular, $x_{i_0-1} R^+ x_{i_0} \wedge x_{i_0} R^+ x_{i_0-1}$, a contradiction since $Q_{i_0} = <_{R^+}$ and $<_{R^+}$ is defined as $x <_{R^+} y \Leftrightarrow x R^+ y \wedge \neg y R^+ x$. Hence, $(<_U \cup <_{R^+})^+$ is irreflexive. ∎

The relation $<^U_{R^+}$ will be called the *proper partial order approximation of the ranking relation R*. If $<_U \subseteq <_{R^+}$ then $<_{R^+} = <^U_{R^+}$.

PROPOSITION 3.4. $\forall x, y \in X.\ x <^U_{R^+} y \Rightarrow \neg\, y R x$.

PROOF. Suppose $\exists x, y \in X.\ x <^U_{R^+} y \Rightarrow y R x$. This means there are $n \geq 1$, $x_0, \ldots, x_n \in X$, $x_0 = x$, $x_k = y$, $Q_1, \ldots, Q_n \subseteq X \times X$, such that

$$x_0 Q_1 x_2 \ldots x_{n-1} Q_n x_0 R x_0,$$

where Q_i is either $<_U$ or $<_{R^+}$. Since $yRx \Rightarrow \neg y <_U x$, then at least one Q_i, say Q_{i_0}, must be equal to $<_{R^+}$, i.e., $Q_{i_0} = <_{R^+}$. But all $Q_i \subseteq R_+$, so we have $\forall i, j \leq n.\ x_i R^+ x_j \wedge x_j R^+ x_i$. In particular, $x_{i_0-1} R^+ x_{i_0} \wedge x_{i_0} R^+ x_{i_0-1}$ which is a contradiction since $Q_{i_0} = <^+_R$ and $<^+_R$ is defined as $x <_{R^+} y \Leftrightarrow x R^+ y \wedge \neg y R^+ x$. ∎

If $<^U_{R^+}$ is a weak order, we can set $<_g = <^U_{R^+}$. The property $\forall x, y \in X.\ x <_g y \Rightarrow \neg y R x$ can be interpreted as a weaker version of the Arrow's Axiom of Binary Relevance (see [2] for details).

If $<^U_{R^+}$ is not a weak order, we need to find a "good" weak order approximation. To do this, we will follow the approach suggested in [7].

Let us assume that set X is believed to be weakly ordered (by, for example, a group ranking relation $<_g$), but the discriminatory power of the data acquisition process, which seeks to uncover this order, is limited. For example, experts may not always know enough about the problem and may make educated guesses or even select choices at random, the number of experts may be insufficient, or instruments may not be precise enough, etc. The acquired data establish only a partial order $<_p$ which is a partial picture of the underlying order. We seek, however, an extension process which is expected to identify correctly the ordered pairs that are not part of the data. To formulate this problem in a formal way, we need a new concept. To simplify the notation, let $<_p \subseteq X \times X$ be a partial order defined as $<_p = <^U_{R^+}$. Let us define $\approx_p \subseteq X \times X$ as follows:

$$x \approx_p y \Leftrightarrow (\forall z \in X.\ z \sim_p x \Leftrightarrow z \sim_p y) \vee x = y.$$

Various properties of $\approx_p$ are analyzed in [7]. We need to recall only one here.

PROPOSITION 3.5. [7]. *A partial order $<_p$ is weak if and only if $x \approx_p y \Leftrightarrow x \sim_p y \vee x = y$.* ∎

It is worthwhile noting that weak order extensions reflect the fact that if $x \approx_p y$, then *all reasonable methods* for extending $<_p$ will have x equivalent to y in the extension, since there is nothing in the data that distinguishes between them (for details, see [7]).

We will say that a weak order $<_w \subseteq X \times X$ is a *proper weak order extension of* $<_p$ if and only if

$$(x <_p y \Rightarrow x <_w y) \quad \text{and} \quad (x \approx_p y \Rightarrow x \sim_w y \vee x = y).$$

If X is finite, then for every partial order $<_p$, there always exists a proper weak extension. If $<_p$ is weak, then its only weak extension is $<_w = <_p$. If $<_p$ if not weak, there is usually more than one such extension. We will examine three such extensions. Let $v_i : X \longrightarrow$ Integers, $i = 1,2,3$, be the following (value) functions for all $x \in X$:

(1) $v_1(x) = |\{y \mid x <_p y\}|$,
(2) $v_2(x) = |\{y \mid y <_p x\}|$,
(3) $v_3(x) = |\{y \mid x <_p y\}| - |\{y \mid y <_p x\}|$.

Let us define $<_{wi} \subseteq X \times X$, $i = 1,2,3$, as follows:

- $\forall x, y \in X.\ x <_{wi} y \Leftrightarrow v_i(x) < v_i(y)$, if $i = 1,3$, and
- $\forall x, y \in X.\ x <_{w2} y \Leftrightarrow v_2(x) > v_2(y)$,

where $<$ is the standard total order of integers.

PROPOSITION 3.6. *For $i = 1,2,3$, the relation $<_{wi}$ is a proper weak order extension of $<_p$.*

PROOF. Since $x <_{wi} y \Leftrightarrow v_i(x) < v_i(y)$, $i = 1,2,3$, then all $<_{wi}$ are weak orders. The following properties imply $<_{wi}$ are also proper extensions. For every finite partial order $<_p$, and every x_1, x_2 : $(|\{y \mid x_1 <_p y\}| \neq |\{y \mid x_2 <_p y\}|) \vee (|\{y \mid y <_p x_1\}| \neq |\{y \mid y <_p x_2\}|) \Leftrightarrow x_1 <_p x_2 \vee$ $x_2 <_p x_1$. ∎

The weak order $<_{w3}$ comes from [7], while v_{w1} and $<_{w2}$ correspond to the left- and right-normal forms for partially commutative monoids (cf. [9]). More algorithms for finding various proper weak extensions are presented in [7]. The choice of an appropriate weak order extension depends on the problem considered. If someone is looking for "the best objects," the order $<_{w1}$ seems to be the appropriate choice. For locating "the troublemakers" (that is, for example, controversial or extreme opinions), the order $<_{w2}$ is more promising, while $<_{w3}$ corresponds to "the safe opinions."

Approximation of partial orders by weak orders is extremely useful in knowledge-based systems because of the weak order extensions. Not only are they easy to implement (usually a count of elements satisfying some conditions), but flexible in terms of fitness to the situation such as stress on precision, cautiousness in making decision, reliability, or even hunt for extremes. What else may a knowledge engineer dream of?

The constructions presented above are independent of the form of R. They can be applied to *any* (irreflexive) relation $R \subseteq X \times X$. Hence, even if the definition of the ranking relation R would change, the results of *this section* would hold. This is important from the knowledge-based system viewpoint, since for some specific cases, the definitions of the ranking relations may vary from what we have proposed in this paper.

4. THE PROPOSED ALGORITHM AND ITS COMPLIANCE WITH ARROW'S CONSISTENCY RULES

Let X be our set of objects and IR $= \{<_1, \dots, <_m\}$ be a set of individual rankings of X. We propose the following algorithm for finding the group ranking $<_g$:

- construct a ranking relation R over IR (as shown in Section 2);
- construct the proper partial order of R as $<^U_{R+}$ (as shown in Section 3);
- if $<^U_{R+}$ is a weak order, then define a group ranking as $<_g = <^U_{R+}$;
- otherwise $<_g = <_w$ where $<_w$ is one of the proper weak order extensions of $<^U_{R+}$.

To illustrate the above algorithm, let us consider eight objects (in this case criteria) $X = \{a, b, c, d, e, f, g, h\}$, four experts, and their individual rankings IR $= \{<_1, <_2, <_3, <_4\}$ (provided by Table 1).

The individual rankings $<_i$'s are, respectively, represented by the following step-sequences:

$$\{a\}\{b\}\{f\}\{c\}\{d\}\{e\}\{h,g\},$$
$$\{d\}\{e\}\{h\}\{g\}\{a\}\{f\}\{c\}\{b\},$$
$$\{b\}\{e\}\{h\}\{g\}\{a\}\{f\}\{c\}\{d\},$$
$$\{a\}\{f\}\{c,d,e,g\}\{h\}\{b\}.$$

Table 1. Rankings provided by the individual experts.

Expert	Criteria							
	a	b	c	d	e	f	g	h
1	1	2	4	5	6	3	7	7
2	5	8	7	1	2	6	4	3
3	5	1	7	8	2	6	4	3
4	1	5	3	3	3	2	3	4

The ranking relation R (defined as a simple majority voting, $R = R_{\text{smv}}$) and its transitive closure R^+ are illustrated in Figure 1. The unanimous preference relation $<_U$, the partial order approximation $<_{R^+}$, and the proper partial order approximation $<^U_{R^+}$ are presented in Figure 2.

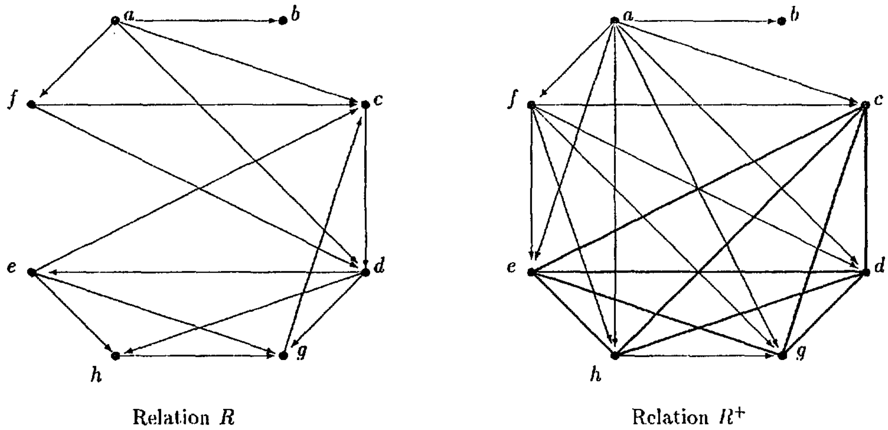

Figure 1. Diagraphs of relations derived from Table 1 (R is assumed to be the simple majority voting, $R = R_{\text{smv}}$).

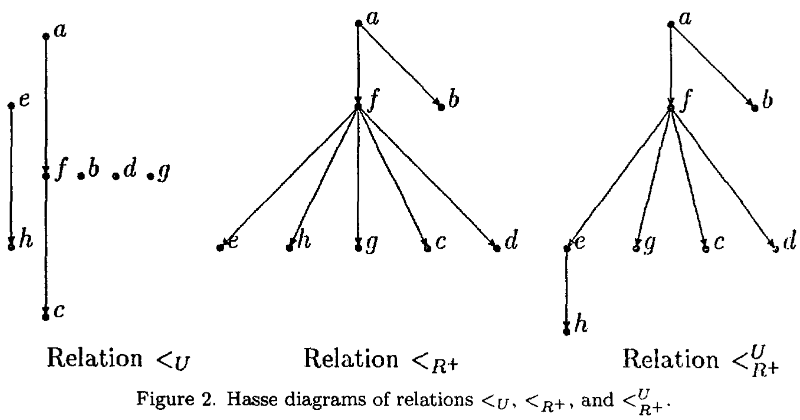

Figure 2. Hasse diagrams of relations $<_U$, $<_{R^+}$, and $<^U_{R^+}$.

Note that R is not transitive, R^+ is not reflexive, $<_U$ is not included in $<^U_{R^+}$, and $<^U_{R^+}$ is not weak. In Figure 2, Hasse diagrams are used to represent the partial orders $<_U$, $<_{R^+}$, and $<^U_{R^+}$. For R, $\bullet x \longrightarrow \bullet y$ means xRy, for R^+, $\bullet x \longrightarrow \bullet y$ means xR^+y, and $\bullet x — \bullet y$ means $xR^+y \wedge yR^+x$.

The value functions $v_1(x)$, $v_2(x)$, and $v_3(x)$ applied to $<^U_{R+}$ give weak orders $<_{w1}$, $<_{w2}$, and $<_{w3}$, which are, respectively, represented by the following step-sequences:

$$\{a\}\{b,f\}\{c,d,e,g\}\{h\},$$
$$\{a\}\{f\}\{e\}\{b,c,d,g,h\},$$
$$\{a\}\{f\}\{b,e\}\{c,d,g\}\{h\}.$$

Table 2 illustrates the three weak extensions of $<^U_{R+}$ constructed in Figure 2. We may define $<_g$ as $<_{wi}$, $i = 1, 2, 3$, or any other proper weak order extension of $<^U_{R+}$.

Table 2. Three weak extensions of the partial order $<^U_{R+}$ in Figure 2.

Rank	$v_1(x)$	x
1	0	a
2	1	b, f
3	2	c, d, e, g
4	3	h

Rank	$v_2(x)$	x
1	7	a
2	5	f
3	1	e
4	0	b, c, d, h, g

Rank	$v_3(x)$	x
1	−7	a
2	−4	f
3	1	e, b
4	2	c, d, g
5	3	h

It is necessary to analyze the compliance of our algorithm with the consistency rules proposed in [1]. The six consistency rules are called, respectively:

(1) weak ordering,
(2) nontriviality,
(3) universal domain,
(4) binary relevance,
(5) Pareto's principle for strict preference,
(6) no dictatorship.

Arrow's Impossibility Theorem proves that *there is no constitution (or social function)* which allows a group ranking $<_g$ to be defined from $\mathrm{IR} = \{<_1, \dots, <_m\}$ in a manner consistent with all of the above listed axioms (1)–(6). The detailed presentation and analysis of the axioms (1)–(6) is in [2]. Axioms (3), (4), and sometimes even (1) may involve controversy (compare [2]) as they fit well in the framework of original applications, but in general they tend to be too strong. They seem to follow from the fact that people are generally quite reluctant to admit any kind of *incomparability* and usually prefer any preference, even if it is a random choice. This phenomenon may be observed to a higher degree in cultures rooted in Greek-Roman tradition which invented the linear (totally ordered) concept of the time continuum. Cultures without this tradition seem to be less reluctant to admit incomparability (fuzziness of the Universe) (also compare [10]). We consider incomparability as a natural phenomenon which together with nondeterminism (see [11] and the closing remarks in *Conclusions* of this paper) really occurs in reality. Axioms (5), (6) (and sometimes (1)) tend to be valid in almost every ranking scheme. Axiom (2) has a cosmetic meaning. Clearly our algorithm is not in contradiction with Arrow's Impossibility Theorem, since we do not assume the validity of all of Arrow's axioms. However, one may note that for a vast majority of practical applications in knowledge-based systems, all important Arrow's axioms are satisfied. This makes our algorithm useful.

THEOREM 4.1. *If* $|X| \geq 2$ *and* $m \geq 3$, *then for every ranking* R, *every proper weak order extension* $<_g$ *of* $<^U_{R+}$ *satisfies Arrow's Axioms (1), (2), (5), and (6).*

PROOF. Axioms (1) and (2) are trivially satisfied. Axiom (5) follows from $<_U \subseteq <^U_{R+} \subseteq <_g$. Axiom (6), "No dictatorship," means that there is no individual whose preferences automatically become the preferences of the group independently of the preferences of the other group members (for a detailed discussion, see [2]). Suppose Expert 1 is a "dictator" and his/her preference is $<_1$.

Let us define the individual ranking $<_i$, $i \neq 1$, and $i \leq m$, as $x <_i y \Leftrightarrow y <_1 x$. Evidently each $<_i$ is a weak order, $<_U$ is empty, and since $m \geq 3$, $x <_1 y \vee \neg xRy$. Note that $R_{\text{smv}} = <_i$, $i \neq 1$, so every ranking relation R satisfies $R \subseteq <_i$, $i \neq 1$, and consequently $R^+ \subseteq <_i$, $i \neq 1$. This also means that R^+ is a partial order, i.e., $R^+ = <_{R^+}$, and that $R^+ \cap <_1 = \emptyset$. Since $<_U = \emptyset$, then $R^+ = <_{R^+} = <^U_{R^+}$, so $<_g = R^+$, i.e., $<_g \cap <_1 = \emptyset$. Because $<_1$ is not empty, therefore, $<_g \neq <_1$, i.e., Expert 1 is not a "dictator." ∎

Axioms (3) and (4), as well as the property $\forall x, y \in X.\ x <_g y \Rightarrow \neg yRx$, a weaker version of Axiom (4), may not be satisfied. For instance, no proper extension of $<^U_{R^+}$ in Table 2 satisfies the property $\forall x, y \in X.\ x <_g y \Rightarrow \neg yRx$.

THEOREM 4.2. *If $|X| \geq 2$ and $m \geq 3$, then for every ranking relation R, the relation $<_g$ defined as $<_g = <^U_{R^+}$ satisfies Arrow's Axioms (2), (5), and (6) and the property $\forall x, y \in X.\ x <_g y \Rightarrow \neg yRx$, a weaker version of Arrow's Axiom (4).*

PROOF. Axiom (2) is trivially satisfied. Axiom (5) follows from $<_U \subseteq <^U_{R^+}$. The proof of Axiom (6) is the same as in Theorem 4.1. The property $\forall x, y \in X.\ x <_g y \Rightarrow \neg yRx$ comes from Proposition 3.4. ∎

5. CONCLUSIONS

There are various ways of formulating priorities: trade-off methods, verbal statements, ratings, and rankings. Voogd [12] reports that the ranking method is preferred by the majority of interviewed experts. Under some circumstances, it might be attractive to use a method that enables experts to express their priorities in a more refined way.

In the authors' opinion, the property $\forall x, y \in X.\ x <_g y \Rightarrow \neg yRx$ is more important than the weak ordering requirement for $<_g$. The requirement that $<_g$ is a partial, or eventually an interval order, seems to suffice when we assume that incomparability exists as natural reality rather than it is a consequence of incomplete knowledge.

Werner Heisenberg in many discussions with Albert Einstein stated that nondeterminism exists in the Universe and that the quantum theory is just a model of nondeterminism. Einstein's position was that all nondeterminism comes from a deficiency of knowledge about the phenomena under consideration. Modern science is inclined to substantiate the validity of Heisenberg's *uncertainty principle* rather than Einstein's position on nondeterminism (see [11]). The uncertainty principle has profound implications for the way in which we view the world. Its relevance to ranking is quite important. It may not be reasonable to insist that everything could or even should be ordered. The assumption of the existence of noncomparable criteria is more natural and practical. In fact, a pairwise comparisons method (see [2,4,5,13,14] utilizes this approach and uses a designated value (usually *one*) for denoting equal importance or lack of knowledge about two compared objects.

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