Mathl. Comput. Modelling Vol. 18, No. 7, pp. 79–84, 1993
Printed in Great Britain.

0895-7177/93 $6.00 + 0.00

A New Definition of Consistency of Pairwise Comparisons

W. W. KOCZKODAJ
Department of Mathematics and Computer Science, Laurentian University
Sudbury, Ontario P3E 2C6, Canada

(Received and accepted February 1993)

Abstract—A new definition of consistency is introduced. It allows us to locate the roots of inconsistency and is easy to interpret. It also forms a better basis than the old eigenvalue consistency for selecting a threshold based on common sense. The new definition of consistency is applicable to expert systems and to knowledge acquisition. It is instrumental in applications of fuzzy sets and the theory of evidence where the definitions of the membership and belief functions are fundamental issues.

A method of pairwise comparison introduced by Thurstone [1] in 1927 was a milestone in decision making science. It is comparable to the introduction of derivatives in calculus or eigenvalues in linear algebra. The decision making process nearly always involves some kind of constituency in modern democratic societies. We have various boards of governors or directors, committees, task groups, city councils, panels of experts, etc. Stormy discussion and various ways of dispute, reasoning, and argumentation take place to arrive at certain decisions. Most constituencies have worked out precise and practical policies for running meetings in an orderly and effective way. What we lack, however, is a device for drawing solid conclusions, and very often the loudest individual wins! Unfortunately, loudness does not necessarily go along with wisdom. Casual thinking does not work well in predicting complex outcomes. Casual thinking is partial, fragmentary, and has no effective way to measure intangibles. In the decision making process, many factors must be considered simultaneously and with about the same degree of importance, therefore an approach with more finesse is necessary to obtain a clear and unambiguous conclusion. It has been shown by numerous examples [2,3] that the pairwise comparison method can always be used to draw the final conclusions in a comparatively easy and elegant way. The brilliance of the pairwise comparison could be reduced to a common sense rule: take two at a time if you are unable to handle more than that.

It is intriguing why such a natural and powerful tool has never become widely accepted by decision makers despite its extreme practicality. The author of this paper truly believes that failure of the pairwise comparison method to become more popular has its roots in the consistency definition. The consistency definition is given by the following formula:

$$\mathrm{cf} = \frac{\lambda_{\mathbf{A}} - \mathrm{order}(\mathbf{A})}{(\mathrm{order}(\mathbf{A}) - 1)\,\lambda_{\mathbf{random}}}, \tag{1}$$

for a reciprocal matrix: $\mathbf{A} = [w_{ij}]$, where w_{ij} is a positive element which expresses the relative importance of two stimuli: i and j.

This project was partially supported by the Natural Science and Engineering Council of Canada under Grant OGP 003838.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Each w_{ij} element has the reciprocal property $w_{ij} = 1/w_{ji}$. For consistent reciprocal matrices, we also have $w_{ij} = w_{ik}/w_{jk}$. In the case of three stimuli, our reciprocal matrix reduces to:

$$\mathbf{R}_3 = \begin{bmatrix} 1 & a & b \\ 1/a & 1 & c \\ 1/b & 1/c & 1 \end{bmatrix},$$

where a expresses a referee's relative preference of stimulus 1 over 2, b expresses preference of stimulus 1 over 3, and c is a relative preference of stimulus 2 over stimulus 3.

We will call matrix $\mathbf{R}_3$ *a basic reciprocal matrix,* since $\mathbf{R}_1$ is a trivial case and $\mathbf{R}_2$ is always consistent. Matrix $\mathbf{R}_3$ is consistent if, and only if, $b = a * c$.

Saaty [4] introduced eigenvalues for handling a gradation in comparative judgments in 1977. Saaty is probably the greatest single contributor to the popularization of the pairwise comparison method. It is no wonder that his heuristic of checking the consistency has not been objectively (or probably even not carefully) looked at. Saaty's consistency is based on 10% of the deviation of the largest eigenvalue of a given matrix from the corresponding eigenvalue of a matrix randomly generated. Its major attraction is universalism: Saaty's consistency definition is good for any order of a matrix. The major drawback of Saaty's consistency definition is the rather unfortunate threshold of 10%. The author of this paper does not believe in rounded numbers like 10% and decided to look at the consistency problem more closely. (His research was dictated by practical problems with consistency of judgments related to ranking hazards of abandoned mines in Canada).

The other major problem is less obvious, but hardly less important. Eigenvalues are surrounded by a certain enigma—not many of us can comprehend the meaning of eigenvalues, but nearly all of us have a certain respect for them, let alone total admiration. In fact, there is clear interpretation of eigenvalues while we have a quite clear view of the consistency of judgments. It will be explained, in the Conclusions, that we do need to have an interpretation of consistency of judgments. This interpretation should be somehow consistent with common sense. The third weakness of Saaty's consistency definition is related to location of inconsistency or rather lack of it. An eigenvalue is a global characteristic of a matrix. By examining it we cannot say which matrix element contributed to the increase of consistency. Costly computations are needed for a matrix of higher than the third order. The above three disadvantages are associated together and a new more intuitive definition of consistency must be introduced.

It is worthwhile to note that for a small (how small is small?) deviation of a matrix, its eigenvalues do not change much, but there is no proof that it works the other way around. We expect that bigger changes in the matrix will cause bigger variations of eigenvalues. However, we cannot be even sure that our reciprocal matrix (which is of a very special shape) will behave this way. Saaty proved that the largest eigenvalue of a reciprocal matrix is equal to the order of the matrix if and only if the matrix is consistent. In lack of any analytic indication that reciprocal matrices behave well as far as eigenvalue change is concern, a brute force method was employed for examining the relationship between consistency and exactness of the solution: all possible cases (4913!) of the reciprocal matrix of the order of three were computed for all judgement values proposed by Saaty from 1 to 9 (and associated with them inverted values $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{9}$). The results showing the relationship between eigenvalues and the accuracy of corresponding solutions are shown in Figure 1.

Figure 1 shows that there is practical justification for using eigenvalues as a consistency measure. However, the threshold of 10% does not look apparent for any "practical" reasons. The 10% deviation of the biggest eigenvalue from the matrix order calculated by (1), as proposed by Saaty limits the solution accuracy to approximately 30%. The solution accuracy is computed as

$$\sqrt{\left(a - \frac{x_1}{x_2}\right)^2 + \left(b - \frac{x_1}{x_3}\right)^2 + \left(c - \frac{x_2}{x_3}\right)^2}, \tag{2}$$

where (x_1, x_2, x_3) is a normalized solution.

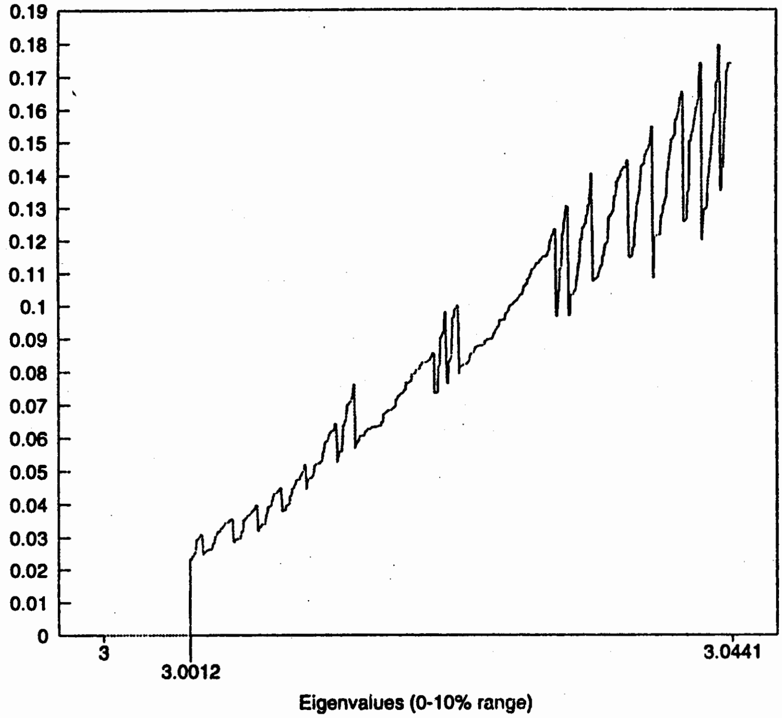

Figure 1. Relationship between eigenvalues and accuracy 0–10% range, as proposed by Saaty.

A careful reader may ask a couple of apparent questions: "Is consistency really so important in the pairwise comparison method?"; "Do we need to look at the consistency of judgments?"

Checking consistency in the pairwise comparison method could be compared in checking that the divisor in a planned division operation is not equal to 0. Simply it does not make sense to divide anything by 0. Similarly all proposed (heuristic) solutions to pairwise comparison models are based on an assumption that the given reciprocal matrix is consistent [4]. Can't we simply assume that the reciprocal matrix is consistent? The answer is definitely not! The power of the pairwise comparison method would be diminished should we ever request that all the judgments are consistent, since we know that most judgements are subjective and nearly always contain some kind of bias. The simplest case of combinatorial consistency was analyzed in [5]. It could be reduced to a case of three stimuli (criteria or attributes) A, B, and C. We assume that $A > B$ and $B > C$, but we also insist on claiming that $C > A$. The definition of a fully consistent reciprocal matrix was introduced in [5], but the only widely accepted measure of inconsistency is due to Saaty.

Our definition of a basic reciprocal matrix $\mathbf{R}_3(a, b, c)$ is based on clear intuition of consistency: it is a measure of deviation from the nearest consistent reciprocal matrix. The interpretation of the consistency measure becomes more apparent when we reduce a basic reciprocal matrix to a vector of three coordinates $[a,\, b,\, c]$. We know that $b = a * c$ holds for each consistent basic reciprocal matrix. We can always produce three consistent basic reciprocal matrices (represented by three vectors) by computing one coordinate from the combination of the remaining two coordinates. These three vectors are: $\left[\frac{b}{c},\, b,\, c\right]$, $[a,\, a * c,\, c]$, and $\left[a,\, b,\, \frac{b}{a}\right]$. The inconsistency measure will be defined as the relative distance to the nearest consistent basic reciprocal matrix represented by one of these three vectors for a given metric. In case of Euclidean (or Chebysheff) metrics, we have:

$$\mathrm{CM} = \min\left(\frac{1}{a}\left|a - \frac{b}{c}\right|,\, \frac{1}{b}\left|b - a\,c\right|,\, \frac{1}{c}\left|c - \frac{b}{a}\right|\right),$$

We can easily extend the above definition to matrices of higher orders. For a given matrix element it can be defined as maximum of CM of all possible triads which include this element. The new relationship is shown in Figure 2.

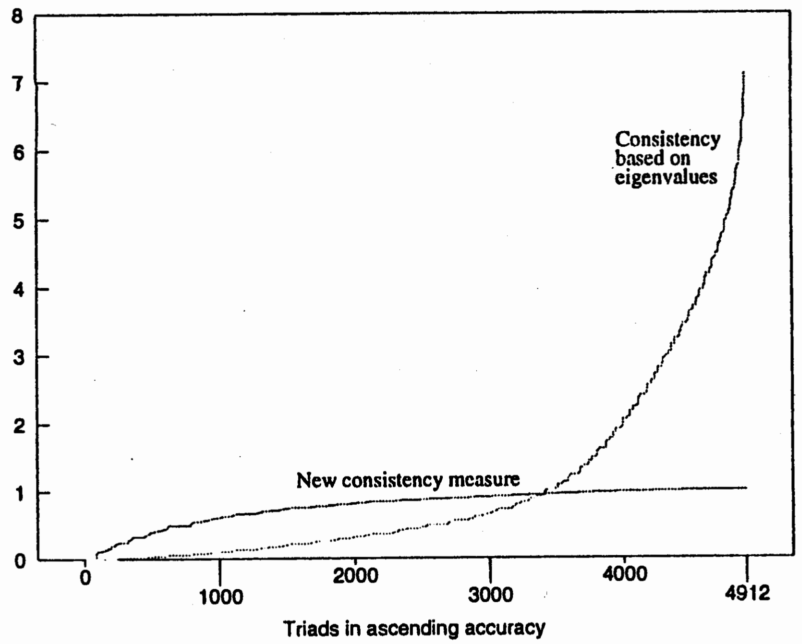

Figure 2. Comparison of the new consistency measure with the consistency based on eigenvalues.

We can easily see that the new consistency measure (CM):

— is easy to interpret (relative deviation from a consistent matrix which may be obtained by keeping two judgments invariant),
— forms a better basis for selecting a threshold based on common sense (e.g., CM of [1, 3, 2] or [3, 6, 3] or [3, 9, 4] or even [2, 8, 3] is 0.25),
— locates the consistency (CM is always associated with a certain matrix element and it is not an enigmatic global matrix characteristic like an eigenvalue).

A very basic question remains: "How is the new consistency definition related to eigenvalues?". The answer to this question is not easy, but the relationship was a surprise even for the author of this paper. With the accuracy of computations, both indicators of consistency remain constant for certain segments or clusters of cases of triads, but the new consistency generates finer segments. For example see Table 1.

The new consistency measure, CM, forms segments with 0.01, 0.2, 0.06, and 0.02 for one corresponding segment of eigenvalues, EV = 3.0012. It is also worthy to note that the accuracy changes with CM rather than with eigenvalues.

It may not be easy to provide a full analytic proof of the above observed regularities, but we can still look at the definition of eigenvalues for a reciprocal matrix. Its characteristic equation is

$$\lambda^3 - 3\lambda^2 + \det(\mathbf{R}_3) = 0,$$

where $\det(\mathbf{R}_3)$ is a determinant of a basic reciprocal matrix which is:

$$\det(\mathbf{R}_3) = \frac{(a\,c - b)^2}{a\,b\,c}.$$

Table 1.

Accuracy	CM	EV
0.024198	0.01	3.0012
0.024198	0.01	3.0012
0.030071	0.01	3.0012
0.030071	0.01	3.0012
0.030924	0.01	3.0012
0.030924	0.01	3.0012
0.02322	0.2	3.0012
0.02322	0.2	3.0012
0.025206	0.06	3.0012
0.025206	0.06	3.0012
0.029384	0.02	3.0012
0.029384	0.02	3.0012
0.025684	0.33	3.0015
0.026003	0.25	3.0015
0.026003	0.25	3.0015
0.026303	0.11	3.0015
0.026303	0.11	3.0015

The largest eigenvalue of $\mathbf{R}_3$ is expressed by the following formula:

$$\lambda_{\max} = 1 + \sqrt[3]{\frac{\det(\mathbf{R}_3)}{2} - \frac{\sqrt{\det(\mathbf{R}_3)}}{2}\sqrt{\det(\mathbf{R}_3) + 4} + 1} + \sqrt[3]{\frac{\det(\mathbf{R}_3)}{2} + \frac{\sqrt{\det(\mathbf{R}_3)}}{2}\sqrt{\det(\mathbf{R}_3) + 4} + 1}\,.$$

(The author does not want to disappoint the reader, but must admit that the above formulas were obtained by Maple, a system for symbolic computation.) It is obvious that the changes of $\lambda_{\max}$ depends on $\det(\mathbf{RM}_3)$ which in turn depends on the changes of $a * c - b$. The new consistency is based on relationship $b = a * c$. Not only does it imply that there is some kind of alignment between Saaty's definition and the new definition of consistency, but it opens new avenues for future ones.

Conclusions

We hope that the new definition of consistency will refocus the attention of researchers from the race of finding better and better approximation of solutions (in forms of heuristics) for inconsistent matrices to devising heuristics which can influence judgments to be more consistent (but by no means totally consistent). Finding an ideal solution for inconsistent (or very inconsistent) matrices is a mirage. It is a theoretically challenging and exciting task but without much use. It could be compared to an attempt at finding lengths of bars measured by a rule which randomly changes its length (by, for example, extreme temperature or atmospheric pressure). The truth is that no "ideal" solution will help us unless we try to understand the source of our problem, which is the inconsistency of judgments. Certainly it is difficult to change the inconsistency unless we know not only its value but the exact location of it. Our definition allows us to locate the inconsistency. It gives the referee a necessary feedback and opportunity of reconsideration of his judgments by using various techniques (e.g., Delphi method). However, it is not advisable to allow the referee the total flexibility. Improvement of his subjective judgment may change to an unsubstantiated race for total consistency of judgments instead of his unbiased subjective opinions. We may, for example, allow the referee to change only a fixed number of opinions by a factor of a fixed total. For example, in case of a matrix of order 4 we have 6 judgments. In this case we may allow a maximum of three modifications with a restriction that the total of all changes does not exceed say 3. In other words each three judgments may be modified by one up or down, or one judgment may be modified by 3 up or down.

The pairwise comparison method is one of the most amazing and universal approaches to solving difficult problems. In particular, contributions to proving superiority of the pairwise comparison method in terms of higher precision of the solution are awaited. They are crucial for convincing (or at least encouraging) practitioners to employ the pairwise comparison method to decision making processes (e.g., regarding environmental problems). It will be of great benefit of all of us.

REFERENCES

1. L.L. Thurstone, A law of comparative judgements, *Psychological Review* **34**, 273–286 (1927).
2. C.-L. Hwang and K. Yoon, *Multiple Attribute Decision Making*, Springer-Verlag, Berlin, (1981).
3. Nijkamp, *Multicriteria Evaluation in Physical Planning*, Springer-Verlag, Berlin, (1991).
4. T.L. Saaty, A scaling methods for priorities in hierarchical structure, *Journal of Mathematical Psychology* **15**, 234–281 (1977).
5. H.A. Davis, *The Method of Pairwise Comparisons*, Griffin, London, (1963).