

Empirical Software Engineering Approach to the Distance-based Inconsistency for Pairwise Comparisons

W.W. Koczkodaj · M. Kosiek ·
J. Szybowski · D. Xu

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Abstract This study presents empirical evidence of the fast convergence of the distance-based inconsistency for pairwise comparisons. It is a follow up of the theoretical proof of the inconsistency convergence. The convergence has been proven by the functional analysis method. As most mathematical proofs, computing the number of iterations has not even been considered. Our empirical research shows that the algorithm converges a fast rate.

Keywords pairwise comparisons · distance-based inconsistency · convergence · knowledge management

1 Introduction

Strict sciences have put more emphasis on processing quantitative (or objective) data than qualitative (subjective) data, which we use more frequently in daily life. However, empirical software engineering often deals with subjective attributes

W.W. Koczkodaj
Computer Science, Laurentian University, 935 Ramsey Lake Road, Sudbury, ON P3E 2C6,
Canada,
Tel.: 705-675-1151 ext.2311
E-mail: wkoczkodaj@cs.laurentian.ca

M. Kosiek
Jagiellonian University, Department of Mathematics and Computer Science, Prof. St.
Łojasiewicza 6, 30-348 Kraków, Poland

J. Szybowski
AGH University of Science and Technology, Faculty of Applied Mathematics, al. Mickiewicza
30, 30-059 Kraków, Poland,
State Higher Vocational School in Nowy Sacz, Staszica 1, 33-300, Nowy Sacz, Poland,
E-mail: szybowsk@agh.edu.pl

D. Xu
Computer Science, Laurentian University, 935 Ramsey Lake Road, Sudbury, ON P3E
2C6, Canada

and evaluations such as software quality or safety. The importance of subjectivity processing is expressed by the idea of *bounded rationality*, proposed by Herbert A. Simon, as an alternative basis for the mathematical modeling of decision making.

Objective data are used more frequently because of the lack of proper methods for processing subjectivity. However, objectivity is often illusive since there is a fine line between objectivity and subjectivity more often than we realize it. For example, when hiring a software engineer, we lack “a yardstick” to measure his/her real software development ability. We compare him/her against other candidates.

Pairwise comparisons allow us to express preferences more easily. These preferences can be highly subjective (e.g., likes/dislikes). Pairwise comparisons were most likely used even before numbers were invented. We can easily envision that “weighting” took place during the Stone Age to decide if a fish, in one hand, can be bartered for a bird in another hand. Ramon Llull was given credit for discovering the Borda count and Condorcet criterion (Llull winner) in the 13th century after the discovery of his lost manuscripts *Ars notandi*, *Ars electionis*, and *Alia ars electionis*, in 2001. However, Condorcet published his voting method based on pairwise comparisons in 1785 in [2] and it is generally assumed to be the first documented use of this method. The next formal use of pairwise comparisons is traced to Fechner in 1860 (see [3]). More recently, pairwise comparisons method was used in [11]. In this journal, pairwise comparisons were applied to software engineering problems in [6, 7] but in a bit different (statistical) way.

It is worth stressing the binary nature of pairwise comparisons. Similarly to binary numbers, pairwise comparisons are practically irreducible since comparisons one object with itself is not really creative. Empirical software engineering often relies on pairwise comparisons (e.g., the bubble sort) without realizing of their use. In fact, every Ω *< condition > then ... else ...* construct is a pair of actions to be selected on the basis of the *< condition >*.

The distance-based inconsistency was introduced in [8]. It was independently analyzed in [1] by practical experimentations. The above publication also stressed that only the distance-based inconsistency is localizing the inconsistencies. In [9], a mathematical (existential) proof of convergence has been provided for the distance-based inconsistency. No empirical study has ever been done and this is the first publication showing how rapid this convergence is in practice.

In software engineering, the consistency concept is particularly important in the design of user interface (often accounting for 50% of all software development costs). Protecting the consistency of databases is one of the main tasks for DBMS.

In [10], the inconsistency axiomatization is proposed. Sometimes, an inconsistency indicator is called a “measure” but it does not have all properties required for a measure defined in mathematical sense. So, the measure is used in the way software engineering uses it.

2 The inconsistency reduction

A distance-based adjective has been used by other researchers for the new inconsistency defined in 1993 in [8]. The distance-based adjective reflects the nature of the inconsistency indicator, which is defined as a minimal distance from the nearest consistent triad in matrix A. Matrix A is defined as:

$$A = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ \frac{1}{a_{12}} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \cdots & 1 \end{pmatrix}$$

In data and knowledge processing, the importance of inconsistency analysis is expressed by the popular adage GIGO (garbage in – garbage out). GIGO summarizes well what has been known for a long time: processing “dirty data” cannot guarantee meaningful results. The distance-based inconsistency localizes the most inconsistent triad (or triads). It is the maximum of all triads a_{ik}, a_{kj}, a_{ij} of elements of A (say, with all indexes i,j,k distinct) of their inconsistency indicators, which in turn are defined as:

$$ii = \min(|1 - \frac{a_{ij}}{a_{ik}a_{kj}}|, |1 - \frac{a_{ik}a_{kj}}{a_{ij}}|). \quad (1)$$

It has been recently simplified to:

$$ii = 1 - \min(\frac{a_{ij}}{a_{ik}a_{kj}}, \frac{a_{ik}a_{kj}}{a_{ij}}). \quad (2)$$

The process of reducing global inconsistency of a pairwise comparisons matrix (PC matrix), is based on the detection of triads (say, $\{a_{ik}, a_{kj}, a_{ij}\}$) with the maximal inconsistency. When such a triad is located, we modify the value of a_{ik} , a_{kj} or a_{ij} in order to make the replaced triad fully consistent. This method was described in [9] and mathematically proven to be convergent when the most inconsistent triad is replaced by the closest triad, according to the Euclidean distance. A mathematical proof was provided that such a process leads to a fully consistent matrix but its rate of convergence has never been examined.

2.1 The distance-based inconsistency reduction algorithm

In practice, we change only one value in a triad. Depending on the application, it may take days or even weeks to call an expert panel, gather data, analyze it, and make a decision about which value is better. Inconsistency analysis allows us to locate the most inconsistent triad. However, in our experimentation, we modify all three values: a_{ik} , a_{kj} , and a_{ij} . This is done by splitting the total modification to three elements of a triad by minimizing the affect of the modification on the initial PC matrix. Let us assume that the most inconsistent triad is $\{a_{ik}, a_{kj}, a_{ij}\}$. According to equation (1):

$$I3 = \min|1 - \frac{a_{ik} * a_{kj}}{a_{ij}}, 1 - \frac{a_{ij}}{a_{ik} * a_{kj}}|$$

The name $I3$ has been selected to denote the triad (hence 3) inconsistency (I). To make this triad consistent ($I3 = 0$), three variables (let us say $\Delta_{ik}, \Delta_{kj}, \Delta_{ij}$) are added to each entry in this triad. The following equation can be obtained when $\Delta_{ik}, \Delta_{kj}, \Delta_{ij}$ meet above requirements:

if $a_{ik} * a_{kj} < a_{ij}$

$$(a_{ik} + \Delta_{ik}) * (a_{kj} + \Delta_{kj}) = (a_{ij} - \Delta_{ij}) \quad (3)$$

where numbers $\Delta_{ik}, \Delta_{kj}, \Delta_{ij}$ are positive.

if $a_{ik} * a_{kj} > a_{ij}$

$$(a_{ik} - \Delta_{ik}) * (a_{kj} - \Delta_{kj}) = (a_{ij} + \Delta_{ij}) \quad (4)$$

where numbers $\Delta_{ik}, \Delta_{kj}, \Delta_{ij}$ are positive.

By assigning three values to Δ_{ik}, Δ_{kj} and Δ_{ij} respectively, the triad will be fully consistent. In order to keep Δ_{ik}, Δ_{kj} and Δ_{ij} relatively small for a_{ik}, a_{kj} and a_{ij} , we assign values to Δ_{ik}, Δ_{kj} and Δ_{ij} according to their relative weights among a_{ik}, a_{kj} and a_{ij} .

Hence, we can come to the following:

$$\Delta_{ik} = \frac{a_{ik}c}{a_{ik} + a_{kj} + a_{ij}}, \Delta_{kj} = \frac{a_{kj}c}{a_{ik} + a_{kj} + a_{ij}}, \Delta_{ij} = \frac{a_{ij}c}{a_{ik} + a_{kj} + a_{ij}} \quad (5)$$

where c is a constant.

Combining equation (3) and equation (5), we can get another equation:

for $a_{ik} * a_{kj} < a_{ij}$:

$$\left(\frac{a_{ik} * a_{kj}}{(a_{ij} + a_{ik} + a_{kj})^2}\right)c^2 + \frac{a_{ij} + 2 * a_{ik} * a_{kj}}{a_{ij} + a_{ik} + a_{kj}}c + a_{ik} * a_{kj} - a_{ij} = 0 \quad (6)$$

for $a_{ik} * a_{kj} > a_{ij}$:

$$\left(\frac{a_{ik} * a_{kj}}{(a_{ij} + a_{ik} + a_{kj})^2}\right)c^2 - \frac{a_{ij} + 2 * a_{ik} * a_{kj}}{a_{ij} + a_{ik} + a_{kj}}c + a_{ik} * a_{kj} - a_{ij} = 0, \quad (7)$$

which are quadratic polynomials.

Recall, that for a polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

its n roots x_1, \dots, x_n satisfy the so-called Vieta's formulas

$$\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1} x_{i_2} \cdot \dots \cdot x_{i_k} = (-1)^k \frac{a_{n-k}}{a_n}. \quad (8)$$

By solving equations (6) and (7), c can be obtained and then all $\Delta_{ik}, \Delta_{kj}, \Delta_{ij}$ can be determined. The discriminant for both equations is the same and is equal to:

$$\begin{aligned} \frac{(a_{ij} + 2a_{ik} * a_{kj})^2}{(a_{ij} + a_{ik} + a_{kj})^2} - 4 \frac{(a_{ik} * a_{kj} - a_{ij}) * a_{ik} * a_{kj}}{(a_{ij} + a_{ik} + a_{kj})^2} &= \\ &= \frac{a_{ij}^2 + 8a_{ij} * a_{ik} * a_{kj}}{(a_{ij} + a_{ik} + a_{kj})^2} > 0, \end{aligned}$$

and it implies that both equations have exactly two solutions c_1 and c_2 .

Furthermore, when $a_{ik} * a_{kj} < a_{ij}$, from the formulas (8), the roots of equation (6) satisfy

$$c_1 * c_2 = \frac{(a_{ik} * a_{kj} - a_{ij}) * (a_{ij} + a_{ik} + a_{kj})^2}{a_{ik} * a_{kj}} < 0,$$

which implies that only one of them is positive. In this case, we take the positive root as the solution. If we selected the negative solution, both $a_{ik} + \Delta_{ik}$ and $a_{kj} + \Delta_{kj}$ would be negative.

When $a_{ik} * a_{kj} > a_{ij}$ the product of roots of equation (7) is given by the same formula, so it is positive. At the same time, again from the formulas (8),

$$c_1 + c_2 = \frac{(a_{ij} + 2 * a_{ik} * a_{kj}) * (a_{ij} + a_{ik} + a_{kj})}{a_{ik} * a_{kj}} > 0,$$

which implies that both of them are positive. In this case we take the smaller value as the answer to this triad. If we took the bigger one, $a_{ik} - \Delta_{ik}$ and $a_{kj} - \Delta_{kj}$ would be negative.

The full proof of convergence, based on the functional analysis, was provided in [9]. To show it more explicitly, let us consider instead each matrix

$$A = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ \frac{1}{a_{12}} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \cdots & 1 \end{pmatrix}$$

the skew symmetric matrix

$$\log A = \begin{pmatrix} 0 & \log a_{12} & \cdots & \log a_{1n} \\ -\log a_{12} & 0 & \cdots & \log a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\log a_{1n} & -\log a_{2n} & \cdots & 0 \end{pmatrix}$$

Denote by $M(a_{ik}, a_{kj}, a_{ij})$ the set of logarithms of all matrices which are consistent with respect to a given triad a_{ik}, a_{kj}, a_{ij} . It follows from formula (2) that $M(a_{ik}, a_{kj}, a_{ij})$ is a linear subspace of the space of all skew symmetric matrices. Moreover, the intersection of all such subspaces is equal to the subspace of all skew symmetric matrices which are logarithms of consistent matrices.

Consider an algorithm which is, for a given triad a_{ik}, a_{kj}, a_{ij} , an orthogonal projection of $\log A$ onto $M(a_{ik}, a_{kj}, a_{ij})$ with respect to some inner product. Such algorithm applied sequentially to a sequence of triads is convergent, as stated in [5] where the proof was incomplete. A complete proof was finally provided in [9]. The choice of an inner product is not important since Theorem 1 in [9] is general enough. Unfortunately, Theorem 1 does not provide any analytical estimation of the number of steps required for the convergence, Monte Carlo simulation is needed for its effectiveness.

Our computations clearly indicate that the convergence takes place in fewer steps than we have anticipated it. As such, it is “good enough” (also known as “satisfying”) for practical applications of pairwise comparisons. As opposed to optimal decisions, satisfying, a portmanteau of satisfy and suffice, is a decision-making strategy that attempts to meet an acceptability threshold. In our case, this threshold for inconsistency has been assumed (as a heuristic) to be 1/3. It is worth noticing that “a satisfying strategy” may often be (near) optimal if the costs of the decision-making process itself are considered as a part of the objective

function. By “costs”, we understand not the financial problem but “other aspects” related to solving our decision problem. It may vary from obtaining the complete information (usually, an impossible task) to assess the impact of our decision on “public safety” or “public acceptance”.

2.2 An example of the weight-based method for the inconsistency reduction

Let us assume we have a triad $\{4.2, 0.7, 1.8\}$. This triad is inconsistent since $4.2 \cdot 0.7 = 2.94 > 1.8$. but $a_{ij} = 2.94$ makes this triad consistent. However, this may cause bigger changes in other triads. After all, the change of a_{ij} from 1.8 to 2.94 is a relatively significant change (nearly doubled). An improvement is expected by finding the corresponding variables $\Delta_{ik}, \Delta_{kj}, \Delta_{ij}$ from this equation: $(4.2 - \Delta_{ik}) * (0.7 - \Delta_{kj}) = (1.8 + \Delta_{ij})$

Let us verify that values of $\Delta_{ik}, \Delta_{kj}, \Delta_{ij}$ are not significantly affecting another triad (or triads). According to equation (5), we get:

$$\begin{aligned}\Delta_{ik} &= c * 4.2 / (4.2 + 0.7 + 1.8) = 4.2c / 6.7 = 0.63c \\ \Delta_{kj} &= c * 0.7 / (4.2 + 0.7 + 1.8) = 0.7c / 6.7 = 0.104c \\ \Delta_{ij} &= c * 1.8 / (4.2 + 0.7 + 1.8) = 1.8c / 6.7 = 0.27c\end{aligned}$$

By solving this equation:

$$(4.2 - 0.63c) * (0.7 - 0.104c) = (1.8 + 0.27c)$$

we get $c_1 = 16.44, c_2 = 1.059$.

According to the earlier discussion, we take c_2 as our solution. Therefore, we have:

$$\Delta_{ik} = 0.67$$

$$\Delta_{kj} = 0.11$$

$$\Delta_{ij} = 0.28$$

hence:

$$a_{ik} = 4.2 - 0.67 = 3.53$$

$$a_{kj} = 0.7 - 0.11 = 0.59$$

$$a_{ij} = 1.8 + 0.28 = 2.08$$

and the new triad is $\{3.53, 0.59, 2.08\}$.

2.3 Approximation of pairwise comparison matrices

The consistent matrices are of special interest since they are generated by a vector of weights. For each given PC matrix A , our algorithm finds a consistent PC matrix B such that:

$$\delta(A, B) \leq \delta(A, C)$$

for every consistent PC matrix C . This means that B minimizes the distance δ from A to consistent PC matrices.

Let $\lambda : \mathcal{R}_+^{\mathcal{J}} \rightarrow \mathcal{R}^{\mathcal{J}}$ and $\mu : \mathcal{R}^{\mathcal{J}} \rightarrow \mathcal{R}_+^{\mathcal{J}}$ be the coordinatewise logarithmic and exponential mappings for $A = (a_{ij})$, $B = (b_{ij})$:

$$\begin{aligned} B = \lambda(A) \quad & \text{iff} \quad b_{ij} = \log(a_{ij}) \quad \text{for every } i, j = 1, \dots, N \\ A = \mu(A) \quad & \text{iff} \quad a_{ij} = \exp(b_{ij}) \quad \text{for every } i, j = 1, \dots, N \end{aligned}$$

Obviously, the functions λ and μ are the inverse of each other. They linearize the pairwise comparison mathematics by translating the consistency conditions $a_{ij} \cdot a_{jk} \cdot a_{ki} = 1$ in space $\mathcal{R}_+^{\mathcal{J}}$ into linear conditions $b_{ij} + b_{jk} + b_{ki} = 0$ in the linear space $\mathcal{R}^{\mathcal{J}}$. As a result, the image of the (non-linear) subspace of all consistent matrices under mapping λ is a linear subspace of \mathcal{R}^N (and the same is true for the image of the subspace of all reciprocal matrices). Thus it is natural to define distance in $\mathcal{R}_+^{\mathcal{J}}$ as follows:

$$\delta(A', A'') = d(\lambda(A'), \lambda(A''))$$

where d is the Euclidean distance in $\mathcal{R}^{\mathcal{J}}$.

The distance δ has appeared in the *Logarithmic Least Square Method (LLSM)*. LLSM solves the approximation problem elegantly. Given an arbitrary $N \times N$ pairwise comparison matrix A , let $B = \lambda(A)$, the best (and unique) approximation $Z \in \mathcal{R}^{\mathcal{J}}$ exists such that Z is logarithmically consistent (meaning that $Y = \mu(Z)$ is consistent). Indeed, Z is the orthogonal projection of B onto the linear space of all logarithmically consistent matrices in $\mathcal{R}^{\mathcal{J}}$. This means that Y is not only the best but also unique consistent approximation of A (with respect to distance δ , of course).

In practice, the mathematical meaning of “best” is understood from the application point of view. It may vary from one application to another and it seems reasonable to take advantage of expert opinions in the process of arriving at a consistent matrix.

2.4 The Convergence of Consistency Approximations

Let \mathbf{L} be a non empty finite family of linear subspaces of \mathcal{R}^N . Let \mathbf{L}^\cap be the family of all intersections of subfamilies of family \mathbf{L} . If U, V are linear subspaces of \mathcal{R}^N , neither of which is contained in the other, we define their separation $\sigma(U, V)$ as the smallest non-zero principal angle between U and V . In other words, $\sigma(U, V)$ is the smallest angle between non-zero vectors $u \in U$ and $v \in V$, both of which are orthogonal to $U \cap V$. By compactness, $\sigma(U, V)$ is always strictly positive. If $U \subseteq V$ or $V \subseteq U$, we set $\sigma(U, V) = \pi/2$. The separation $sep(\mathbf{L})$ of the family \mathbf{L}

is defined as the minimum of separations $d(V, V')$ taken over all pairs $V, V' \in \mathbf{L}^\cap$ such that $d(V, V') > 0$ – but if $d(V, V') = \frac{\pi}{2}$ (i.e. $V \subseteq V'$ or $V' \subseteq V$) for every $V, V' \in \mathbf{L}^\cap$ then we define $sep(\mathbf{L}) = 0$. Note that $sep(\mathbf{L}) = 0$, if and only if \mathbf{L} is ordered linearly with respect to the inclusion relation \subseteq . For this reason, the consistency approximation is trivially true for $sep(\mathbf{L}) = 0$. Let $p_V(y)$ be the orthogonal projection of point y onto the linear subspace V , for every $y \in \mathcal{R}^N$ and for every linear subspace V of \mathcal{R}^N .

In [5], the notion of separation $sep(\mathbf{L})$ resulted in proving the following Theorem 1.

Theorem 1 *Let \mathbf{L} be a non empty finite family of the linear subspaces of \mathcal{R}^N . Let $W = \bigcap \mathbf{L}$ be the intersection of members of \mathbf{L} , let $x \in \mathcal{R}^N$ and $y = p_W(x)$. Furthermore, let $w : \{1, 2, \dots\} \rightarrow \mathbf{L}$ be a sequence such that for every $V \in \mathbf{L}$ equality $V = w(n)$ holds for infinitely many different $n = 1, 2, \dots$. Define $x_0 = x$ and $x_n = p_{w(n)}(x_{n-1})$ for every $n > 0$. Then $\lim_{n \rightarrow \infty} x_n = y$.*

Our computing experimentation demonstrates that the weight-based inconsistency reduction algorithm can efficiently reduce the global inconsistency of a “Not-so-inconsistent” (NSI) PC matrix to a certain threshold value (1/3 is usually considered as the acceptable inconsistent level for most applications). The initial PC matrix is not expected to be fully consistent. Solving real-life problems usually involves inconsistent assessments. However, a matrix with large inconsistency is undesirable according to “garbage in, garbage out (GIGO)” principle. Inconsistencies often reflect assessing “every criterion being more important than another”.

The concept of an NSI PC matrix was introduced in [4] by the first author of this study. Monte Carlo experiments in [4] demonstrated (on the basis of 1,000,000 cases) no statistical difference between the geometric means and eigenvalue methods of computing weights. A randomly selected deviation was applied to elements of a fully consistent matrix rendering it inconsistent. The same method is also used in this study. For an inconsistency to occur, a minimum size of 3 for PC matrix is required since at least one triad needs to exist. Needless to say that for two comparisons, inaccuracy (not inconsistency) takes place. We use $n = 7$ as the maximal PC matrix size. For a matrix with n elements, there are $n*(n - 1)/2$ comparisons. It gives us 21 comparisons for $n = 7$ and it is a psychological limit for most respondents to cooperate (we wonder who would agree to compare 100 objects giving 4950 pair combinations?) Fig. 1 shows all the existing triads for $n = 7$. Nodes in the graph in Fig. 1 are indexes of triad elements.

3 The relationship of deviation and maximal inconsistency

We produce not-so-inconsistent (NSI) PC matrices by using a random deviation $\Delta > 0$. For $\Delta = 0$, the PC matrix, generated from a random vector with positive coordinates, is fully consistent. By increasing Δ , the inconsistency of the PC matrix is also expected to increase. In order to examine the relationship between Δ and maximal inconsistency, we follow this algorithm:

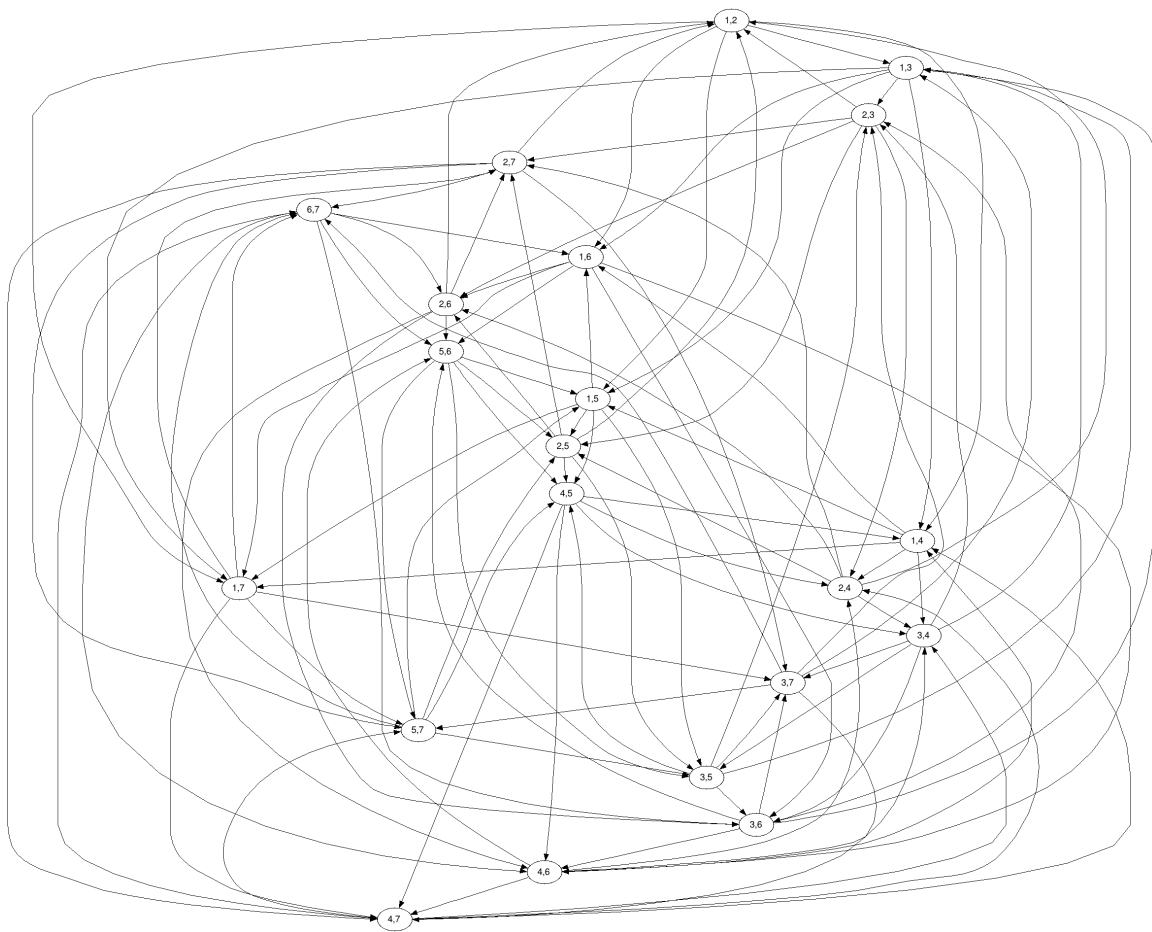


Fig. 1 All triads existing in a 7 by 7 PC matrix

1. Generate random PC matrices,
2. Adjust the deviation of each matrix from 0 to 0.5 with increasing 0.0005 each iteration,
3. Record the maximal inconsistency of 1,000 matrices for each deviation,
4. Compute the average maximal inconsistency of 1,000 matrices for each deviation.

The NSI PC matrix is obtained by:

1. Randomize a vector (say $V^{[*]}$)
2. Generate the fully consistent PC matrix (say A) by $a_{ij} = v_i/v_j$.

Fig. 2 shows the histogram of inconsistency in a NSI PC matrix generated by adding random generated deviation from 0 to 0.5. Evidently, it is a normal distribution as was expected.

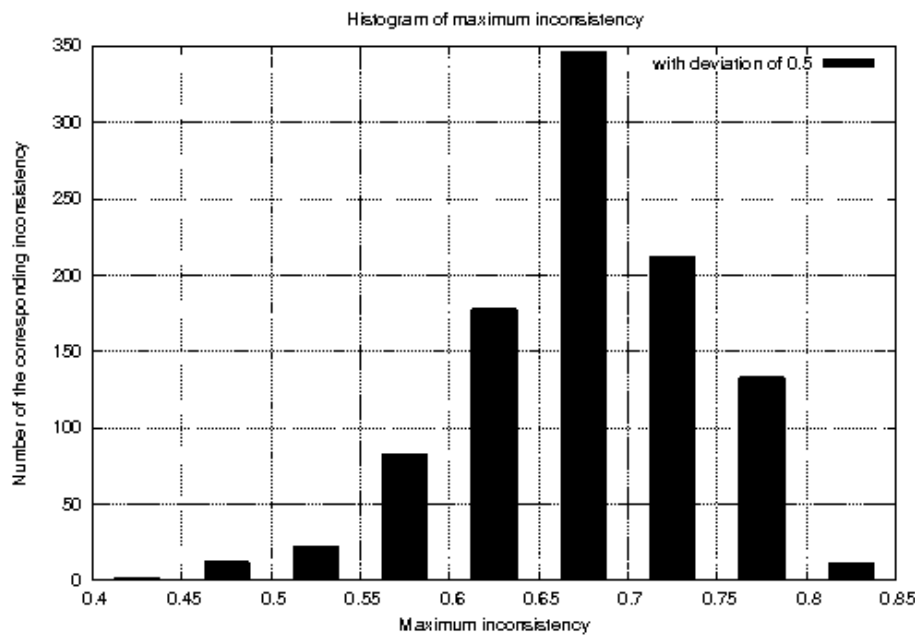


Fig. 2 Histogram of Inconsistency

Fig. 3 shows the result of the relation between deviation and inconsistency:

As we can see from Fig. 3, the maximal inconsistency increases with the deviation. It is nearly linear (but not quite) dependency for 1,000 generated NSI PC matrices. The maximal inconsistency is still below 0.7, since the deviation was not significantly high (between 0 and 0.5)

4 The weight-based inconsistency reduction

In order to test the convergence of weight-based method of the inconsistency reduction, we:

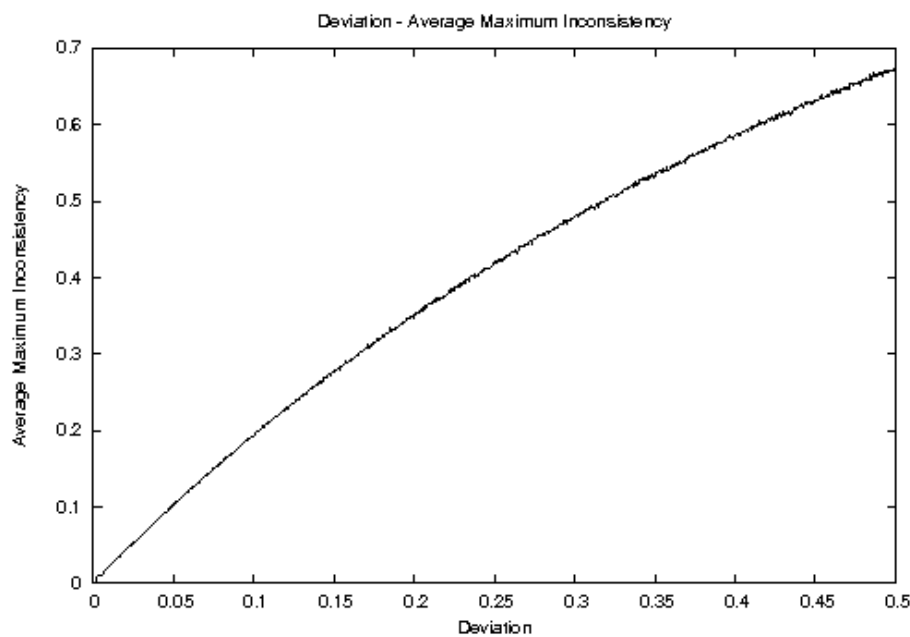


Fig. 3 Dependence of maximal inconsistency and the element deviation

1. Generate random 7 by 7 NSI PC matrices as described above.
2. Add a deviation of 0.5 on each entry in the upper PC matrix triangle.
3. Record the maximal inconsistency of each PC matrix.
4. Count the number of triads with the inconsistency larger than $1/3$.
5. Count iterations needed to reduce the maximal inconsistency to a maximal value of $1/3$.

Fig. 4 shows the histogram of numbers of iterations needed to reduce the inconsistency to equal or less than $1/3$: Fig. 4 shows the histogram of the number of iterations needed for the inconsistency reduction to reach the acceptable level of $1/3$ after 4, 5, 6, or 7 iterations. To analyze the data in details, we compute the average number of iterations for maximal inconsistency for each NSI PC matrix. The relationship between maximal inconsistency and average numbers of iterations is shown by Fig. 5. It is encouraging to see that not more than seven iterations are needed. Fig. 5 shows the number of iterations needed to bring the inconsistency under the required level of acceptance $1/3$. The number of iterations actually depends more on the number of triads with an inconsistency larger than $1/3$ than on one triad with a high inconsistency.

Fig. 6 shows the relation between the number of iterations needed to bring the inconsistency under the required level of acceptance $1/3$ for the given number of inconsistent triads.

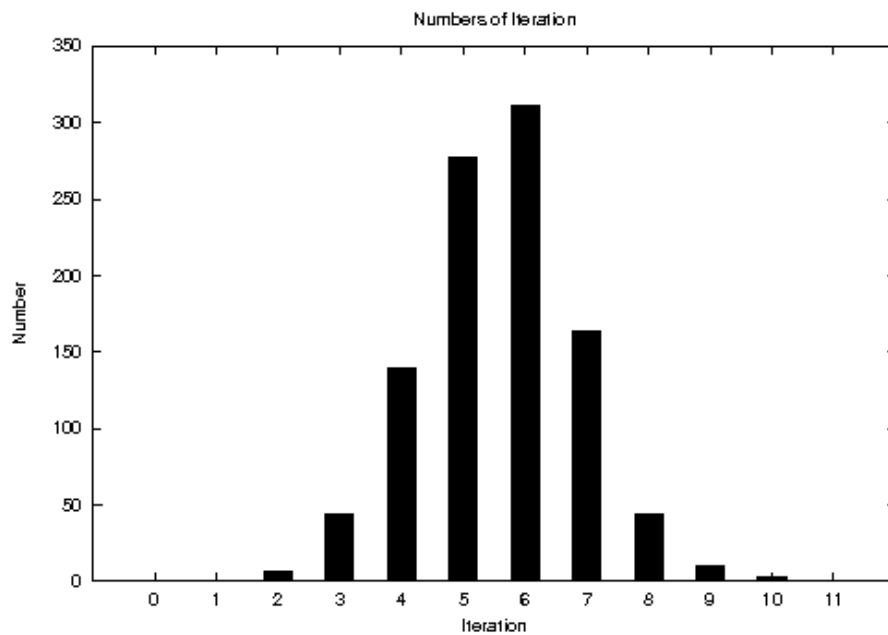


Fig. 4 Histogram showing the numbers of iterations needed to reduce the inconsistency to not more than $1/3$

Fig. 6 shows that the number of triads with large inconsistency affects the number of iterations needed to reduce the inconsistency to the acceptable level $1/3$.

5 Conclusions

We have generated 1,000 NSI PC matrices with ranks ranging from 4 by 4 to 7 by 7. The convergence rate was rapid. Bringing matrices to an inconsistency below $1/3$ takes place usually in no more than 10 steps for the worst randomly generated case. The inconsistency problem in pairwise comparisons is the most fundamental problem. Intensive literature searches strongly support that computing the exact number of iterations needed for the inconsistency reduction to the acceptable level has not been done yet.

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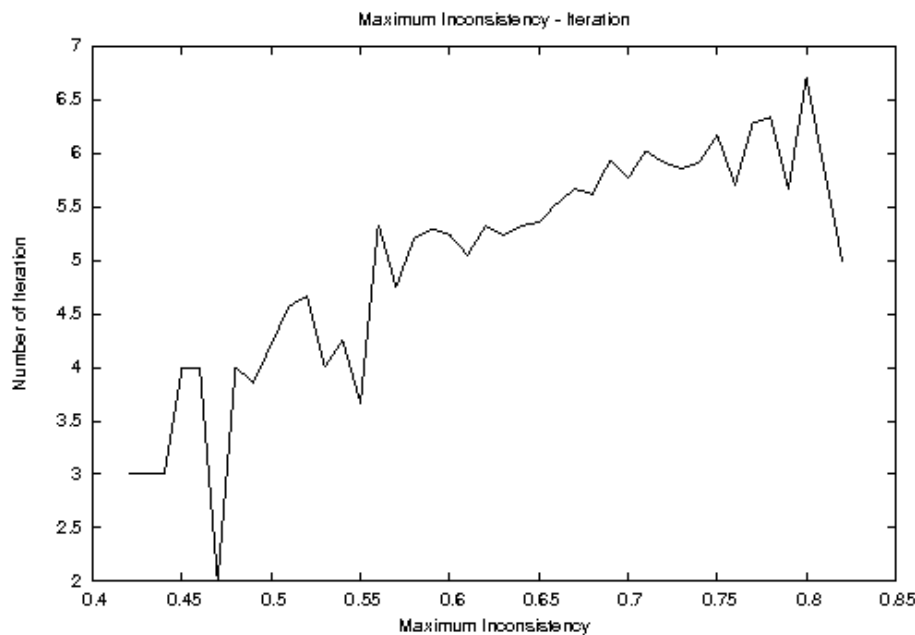


Fig. 5 The number of iterations needed to bring the maximal inconsistency of the PC matrix under $1/3$

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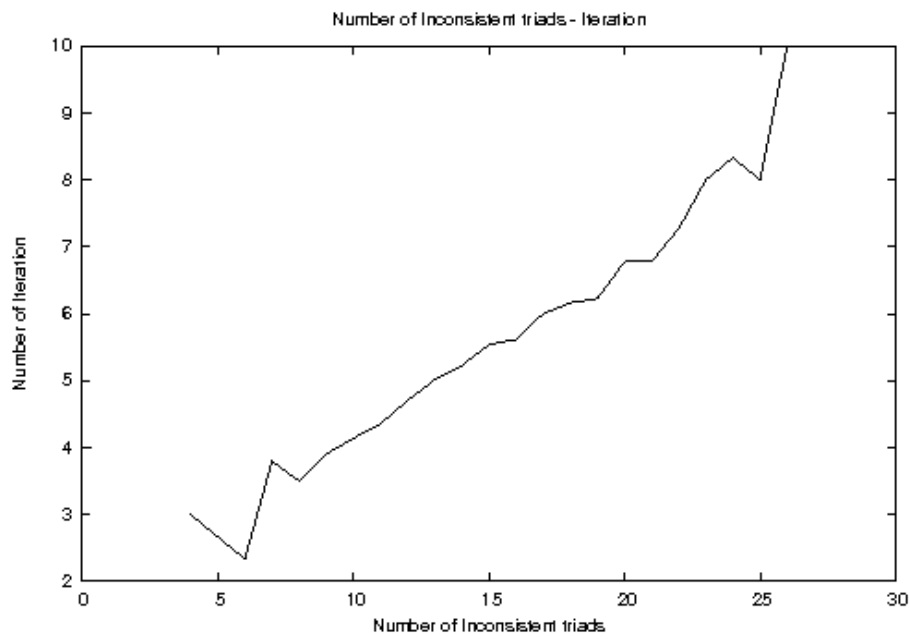


Fig. 6 The relationship between the number of triads with inconsistency greater than $1/3$ and the average number of iterations