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An Orthogonal Basis for Computing a Consistent Approximation to a Pairwise Comparisons Matrix

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Abstract—The algorithm for finding a consistent approximation to an inconsistent pairwise comparisons matrix is based on a logarithmic transformation of a pairwise comparisons matrix into a vector space with the Euclidean metric. An orthogonal basis is introduced in the vector space formed by the images of consistent matrices. The required consistent approximation is the orthogonal projection of the transformed matrix onto this space.

Keywords—Approximation algorithm, Pairwise comparisons, Orthogonal basis, Comparative judgments, Inconsistency, Logarithmic transformation.

1. BASIC CONCEPTS OF PAIRWISE COMPARISONS

Triad inconsistency was introduced in [1] and the convergence of triad-based algorithms for reducing inconsistency was proved in [2]. The reader's familiarity with [2] is assumed due to space limitations. Only the most essential concepts of the pairwise comparison method are recalled here. The method of pairwise comparisons was introduced in embryonic form by Fechner (see [3]) and after considerable extension, made popular by Thurstone (see [4]). It can be used as a powerful inference tool and knowledge acquisition technique in knowledge-based systems and data mining. An $n \times n$ *pairwise comparisons matrix* is defined as a square matrix $M = [m_{ij}]$ such that $m_{ij} > 0$

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX for every $i, j = 1, \dots, n$,

$$M = \begin{vmatrix} 1 & m_{12} & \cdots & m_{1n} \\ \frac{1}{m_{12}} & 1 & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{m_{1n}} & \frac{1}{m_{2n}} & \cdots & 1, \end{vmatrix},$$

where m_{ij} expresses an expert's relative preference of stimuli s_i over s_j.

A pairwise comparison matrix M is called *reciprocal* if $m_{ij} = (1/m_{ji})$ (then automatically $m_{ii} = 1$). A pairwise comparison matrix M is called *consistent* if $m_{ij} \cdot m_{jk} = m_{ik}$ for every $i, j, k = 1, \dots, n$. While every consistent matrix is reciprocal, the converse is false in general. Consistent matrices correspond to the ideal situation in which there are exact values $w_1, \dots, w_n$ for the stimuli. The quotients $m_{ij} = w_i/w_j$ then form a consistent matrix.

An important step in the pairwise comparisons inference theory is a theorem (see [5]) which states that for every $n \times n$ consistent pairwise comparisons matrix $M = [m_{ij}]$ there exist positive real numbers $w_1, \dots, w_n$ such that $m_{ij} = w_i/w_j$, for every $i, j = 1, \dots, n$. The vector $w = [w_1, \dots, w_n]$ is unique up to a multiplicative constant. The challenge to the pairwise comparisons method comes from the lack of consistency of the pairwise comparisons matrices which arise in practice (while as a rule, all of the pairwise comparisons matrices are reciprocal). Given an inconsistent $n \times n$ pairwise comparisons matrix M, the theory attempts to provide a consistent $n \times n$ matrix C which differs from matrix M "as little as possible" according to an assumed metric. Algorithms for reducing the triad inconsistency in pairwise comparisons can be simplified by orthogonal projections. The triad inconsistency definition (introduced in [1]) provides an opportunity for reducing the inconsistency of the experts' judgements. It can also be used for data validation in the knowledge acquisition process, one of the fundamental processes in knowledge-based systems, as well as in data mining. The inconsistency measure of a comparison matrix can serve as a measure of the validity of the knowledge.

2. TRIAD L-CONSISTENCY AND ORTHOGONALIZATION

Let us recall that the matrices in the original space consist of positive elements. The problem of the best approximation to a given matrix $M = [m_{ij}]$ by a consistent matrix is transformed into a similar problem of approximating a matrix $M' = [\log m_{ij}]$ by a logarithmic image of a consistent matrix. The benefit of such an approach is that the logarithmically transformed images of consistent matrices form a linear subspace L in $R^{n \times n}$. Each matrix in the subspace L is called a triad L-consistent matrix $M' = [m'_{ij}]$ and satisfies the condition $m'_{ik} + m'_{kj} = m'_{ij}$ for every $i, j, k = 1, \dots, n$. It is much easier to work with linear spaces and use the tools of linear algebra than to work in manifolds (topological or differential). Also, the notion of closeness of matrices is translated from one space to the other since the logarithmic transformation is homeomorphic (a one-to-one continuous mapping with a continuous inverse; see [6, Volume II, p. 593] for details). In other words, two matrices are close to each other in the sense of the Euclidean metric if their logarithmic images are also close in the Euclidean metric.

The approximation problem is thus reduced to the problem of finding the orthogonal projection of the matrix M' on L, since we opt for the least square approximation in the space of logarithmic images of matrices. The following algorithm is proposed to solve the above problem.

1. Find a basis in L.
2. Orthogonalize it.
3. Compute a projection M'' of M' on L using the orthogonal basis of L found in Step 2.

Steps 1 and 2 produce a basis of the space L. A Gram-Schmidt orthogonalization procedure can be used for constructing an orthogonal basis in L (for example, see [6, Volume II, pp. 347–348]). This is done once only for a matrix of a given size n. The presented algorithm for finding a triad L-consistent approximation is based on Step 3; therefore, the approximation problem is reduced to *given a matrix M' (a logarithmic image of the matrix to be approximated by a consistent matrix), find the orthogonal projection M'' of M' onto L.*

The most natural way of solving this problem is to project the matrix M' on the one-dimensional subspaces of L generated by each vector in the orthogonal basis of L and then sum these projections. While most of the computation is routine, the problem of finding an orthogonal basis in the space L is somewhat challenging. For every $n \times n$ consistent matrix M there exists a vector of weights $[w_1, w_2, \dots, w_n]$, unique up to a multiplicative constant such that $m_{ij} = (w_i/w_j)$. One may, thus, infer that the dimension of the space L is $n-1$. As a consequence, this observation stipulates that the space L must have a basis comprised of $n-1$ elements. Analysis of numerous examples has led to the discovery of the following basis matrices $B_k = [b_{ij}^k]$:

$$b_{ij}^k = \begin{cases} 1, & \text{for } 1 \le i \le k < j \le n, \\ -1, & \text{for } 1 \le j \le k < i \le n, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Figure 1 illustrates the basis matrices for $n = 4$. In essence, each basis matrix B_k contains two square blocks of 0s (situated symmetrically about the main diagonal) of size k and $n-k$, a block of 1s of size k by $n-k$ above the main diagonal, and a block of -1s of size $n-k$ by k below the main diagonal, where $k = 1, \dots, n-1$.

$$B_1 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} \qquad B_2 = \begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{vmatrix} \qquad B_3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{vmatrix}$$

Figure 1. An example of a basis of L for $n = 4$.

PROPOSITION 1. *The matrices B_k are linearly independent.*

PROOF. The rank of the following matrix containing as its rows the *enlisted* (by rows) matrices B_k is $n-1$, because the determinant of the submatrix formed by column $12, 13, \dots, 1n$ is equal to 1:

	11	12	13	...	1n	21	22	23	...	n1	n2	n3	...	nn
B_1	0	1	1	...	1	−1	0	0	...	−1	0	0	...	0
B_2	0	0	1	...	1	0	0	1	...	−1	−1	0	...	0
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...
B_{n-1}	0	0	0	...	1	0	0	0	...	−1	−1	−1	...	0

PROPOSITION 2. *In an antisymmetric matrix $B = [b_{ij}]$, the set of conditions*

(i) $b_{pq} + b_{qs} = b_{ps}$, *where p, q, s are pairwise different*

is equivalent to

(ii) $b_{ij} + b_{jl} = b_{il}$, *where $i < j < l$.*

PROOF. Let us assume that s is between p and q. If $p < q$, then (i) can be written as

$$b_{pq} = b_{ps} - b_{qs},$$

and by symmetry

$$b_{pq} = b_{ps} + b_{qs},$$

which is exactly (ii) if we set $(p, s, q) = (i, j, l)$. The reasoning in other cases (for p or q in the middle) is the same because of the symmetry of Condition (i) with respect to the coefficients (p, q, s).

Let us now check if the proposed basis matrices are triad L-consistent. As a consequence of Proposition 2, it is sufficient to check that they are triad L-consistent with respect to the entries above the diagonal.

PROPOSITION 3. *All matrices B_k satisfy the following condition:*

$$b_{ij} + b_{jl} = b_{il}, \qquad \text{where } i < j < l \tag{2}$$

(where superscript k is assumed for each b_{ij}).

PROOF. There are two cases to be considered

(i) $b_{il} = 1$,
(ii) $b_{il} = 0$.

In Case (i), from formula (1), we can infer that $i < k < l$, and therefore, need to consider the following two subcases depending on the position of j relative to k:

- if $j < k$, then $b_{jl} = 1$ and $b_{ij} = 0$,
- if $k < j$, then $b_{jl} = 0$ and $b_{ij} = 1$.

In both subcases, $b_{ij} + b_{jl} = b_{il}$. In Case (ii), in view of our assumption (that is, $i < j < l$), we can see that if $l \le k$, then $b_{ij} = 0$ and $b_{jl} = 0$ because $i, j \le k$ and $j, l \le k$. When $k \le i$, then $k \le j, l$ and also $b_{il} = 0$ and in both situations, the equality $b_{ij} + b_{jl} = b_{il}$ is proven. This completes the proof.

PROPOSITION 4. *A linear combination of triad L-consistent matrices is triad L-consistent.*

PROOF. This follows from elementary algebra. A linear combination of objects satisfying the linear condition in Proposition 3 satisfies the same condition, i.e.,

$$\text{if } x_{ij} + x_{jk} = x_{ik} \text{ and } y_{ij} + y_{jk} = y_{ik},$$
$$\text{then } (\alpha x_{ij} + \beta y_{ij}) + (\alpha x_{jk} + \beta y_{jk}) = \alpha x_{ik} + \beta y_{ik}.$$

The above considerations lead to the following theorem.

THEOREM. *Every triad L-consistent $n \times n$ matrix is a linear combination of the basis matrices $B_k = [b_{ij}^k]$ for $k = 1, 2, \ldots, n-1$, i.e., the matrices B_k span the space L.*

3. THE ORTHOGONALIZATION ALGORITHM

The orthogonal projection of any matrix onto a subspace L is easy to compute if we have an orthogonal basis of L. The Gram-Schmidt orthogonalization process (see, for example, [6, pp. 347–348]) can be used to construct an orthogonal basis. The fairly "regular" form of the basis matrices suggests that an orthogonal basis should also be quite regular. Indeed, by solving $n-2$ systems of linear equations we obtain the following formula for the orthogonal basis matrices T_k (for details, see the Appendix A):

$$T_k = B_k - \frac{n-k}{n-k+1} B_{k-1}, \qquad \text{where } B_0 \text{ is a matrix with all zero elements.}$$

The orthogonal basis for the case $n = 4$ is presented in Figure 2.

$$T_1 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} \qquad T_2 = \begin{vmatrix} 0 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & 0 & 1 & 1 \\ -\frac{1}{3} & -1 & 0 & 0 \\ -\frac{1}{3} & -1 & 0 & 0 \end{vmatrix} \qquad T_3 = \begin{vmatrix} 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \\ -\frac{1}{2} & -\frac{1}{2} & -1 & 0 \end{vmatrix}$$

Figure 2. An example of an orthogonal basis for $n = 4$.

For an arbitrary n one can produce, using formula (3), the $n-1$ matrices T_k, which constitute an orthogonal basis for the space L. The Euclidean norms of the basis matrices can be computed by the following formula:

$$|T_k|^2 = 2\left\{(k-1) * \left[\frac{n-k}{(n-k+1)^2} + \frac{(n-k)^2}{(n-k+1)^2}\right] + n - k\right\} = \frac{2n(n-k)}{(n-k+1)}.$$

Once the matrices T_k are determined, we may compute the following values for a given matrix M' (note that the operation $\cdot$ is a dot product, not a regular matrix product):

$$t_k = \frac{T_k \cdot M'}{|T_k|^2}, \qquad \text{for } k = 1, \ldots, n-1. \tag{5}$$

Viewed geometrically, the values t_k are proportional to the projections of M' along the directions T_k. The algorithm for finding a consistent approximation to the pairwise comparisons matrix by means of an orthogonal basis can be formally specified as follows.

ALGORITHM *OB4PC*:

Input: a reciprocal matrix M of size n by n.

Output: a matrix C, the consistent approximation to M (optimal in the sense of the least square criterion in a linear space after the logarithmic transformation).

Processing steps.

1. Produce matrix $M' = \log M = [\log m_{ij}]$.
2. Generate $n-1$ basis matrices T_k with entries computed according to formula (3).
3. Compute norms $|T_k|$ and coefficients t_k according to formulas (4) and (5), respectively.
4. Compute matrix M'' as the linear combination of basis matrices T_k with coefficients t_k

$$M'' = \sum_{k=1}^{n-1} t_k \times T_k,$$

   where the operation $\times$ is a scalar multiplication. The result is the required projection of M' onto L.
5. Compute $C = \log^{-1} M'' = [\log^{-1} m''_{ij}]$.

END ALGORITHM

The complexity of computing the coefficients t_k, and hence, the matrix C is $O(n^2)$.

4. A NUMERICAL EXAMPLE

Let us now consider the following pairwise comparison matrix which arose in a geological study (see [7]) involving mineral potential assessment (with four criteria: stratigraphy, lithology, alteration/mineralization, and structure).

$$M = \begin{vmatrix} 1 & 1\frac{1}{2} & 2 & 3 \\ \frac{2}{3} & 1 & 3 & 3 \\ \frac{1}{2} & \frac{1}{3} & 1 & 2 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & 1 \end{vmatrix}.$$

It is easy to see that matrix M is not consistent since, for example, $m_{1,2} * m_{2,3} \neq m_{1,3}$ (that is, $1(1/2) * 3 \neq 2$) but it is reciprocal. Finding a consistent approximation (closest in the sense of the assumed metric) is necessary for refining the initial assessments. It is a first step in the consistency-driven approach to the pairwise comparisons method. Let us now trace the algorithm $OB4PC$ on the above example.

Step (1) of algorithm $OB4PC$ produces matrix

$$M' = \begin{vmatrix} 0 & 0.405 & 0.693 & 1.099 \\ -0.405 & 0 & 1.099 & 1.099 \\ -0.693 & -1.099 & 0 & 0.693 \\ -1.099 & -1.099 & -0.693 & 0 \end{vmatrix}$$

(all results are rounded to 0.001 but the computations are carried out in double precision).

Step (2) produces the matrices T_k which were computed in Section 3.

Step (3) produces the norms $[|T_k|] = [2.449, 2.309, 2]$, and the coefficients $[t_k] = [0.732, 0.947, 0.448]$.

Step (4) results in

$$M'' = \begin{vmatrix} 0 & 0.101 & 0.824 & 1.272 \\ -0.101 & 0 & 0.723 & 1.171 \\ -0.824 & -0.723 & 0 & 0.448 \\ -1.272 & -1.171 & 0.448 & 0 \end{vmatrix}.$$

Step (5) gives

$$C = \begin{vmatrix} 1 & 1.107 & 2.280 & 3.568 \\ 0.904 & 1 & 2.060 & 3.224 \\ 0.439 & 0.485 & 1 & 1.565 \\ 0.280 & 0.310 & 0.639 & 1 \end{vmatrix},$$

easily verified to be a consistent reciprocal matrix (to within rounding error).

5. CONCLUSIONS

Until now, a consistent approximation to an inconsistent pairwise comparisons matrix has been constructed by first computing weights $[w_i]$ and then forming the matrix of quotients $[w_i/w_j]$. Several methods are available for finding the weights (see [5,8] for details) but the issue of their relative superiority is still unresolved (this controversy was, however, addressed in [9]). The algorithm presented here computes the optimal consistent approximation (in the sense of the least square criterion in the linear space resulting from a logarithmic transformation) without finding the weights. The use of an orthogonal basis simplifies the computation of the logarithmic image of the consistent approximation to a given matrix since it is just a linear combination of the basis matrices which need to be computed only once for a problem of a given size. The use of an orthogonal basis leads to an algorithm that is simple to implement (especially in a language supporting matrix operations such as APL2).

APPENDIX

THE DERIVATION OF THE ORTHOGONAL BASIS

Formula (3) can be obtained as follows. Given matrices $B_1, B_2, \dots, B_{n-1}$, find $B'_1, B'_2, \dots, B'_{n-1}$ which are pairwise orthogonal and satisfy the following system of equations:

$$\begin{aligned} B'_1 &= B_1, \\ B'_2 &= a_{12}B_1 + B_2, \\ B'_3 &= a_{13}B_1 + a_{23}B_2 + B_3, \\ B'_4 &= a_{14}B_1 + a_{24}B_2 + a_{34}B_3, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ B'_{n-1} &= a_{1,n-1}B_1 + a_{2,n-1}B_2 + \dots + B_{n-1}. \end{aligned}$$

The following properties of the above basis are helpful in the normalization:

$$|B_i|^2 = 2i(n-i) \text{ and } B_i * B_j = 2i(N-j), \qquad \text{if } i < j.$$

Multiply the second equation by B_1 (dot-product)

$$0 = B_2' B_1 = a_1 2 B_1^2 + B_2 B_1 = a_{12} 2(n-1) + 2(n-2),$$

hence,

$$a_{12} = -\frac{n-2}{n-1}.$$

Multiply the third equation by B_1 and by B_2 (dot-product)

$$0 = B_3' B_1 = a_{13} B_1^2 + a_{23} B_2 B_1 + B_3 B_1 = a_{13} 2(n-1) + a_{23} 2(n-2) + 2(n-3),$$
$$0 = B_3' B_2 = a_{13} B_1 B_2 + a_{23} B_2^2 + B_3 B_2 = a_{13} 2(n-2) + a_{23} 2 * 2(n-2) + 2 * 2(n-3).$$

Solving that system of two equations, we get $a_{13} = 0$ and $a_{23} = -(n-3)/(n-2)$.

Let us consider the equation for B_4. Multiplying it by B_1, B_2, B_3, we get

$$\begin{aligned} 0 &= B_4' B_1 = a_{14} B_1^2 + a_{24} B_2 B_1 + a_{34} B_3 B_1 + B_4 B_1 \\ &= a_{14} 2(n-1) + a_{24} 2(n-2) + a_{34} 2(n-3) + 2(n-4), \\ 0 &= B_4' B_2 = a_{14} B_1 B_2 + a_{24} B_2^2 + a_{34} B_3 B_2 + B_4 B_2 \\ &= a_{14} 2(n-2) + a_{24} 2 * 2(n-2) + a_{34} 2 * 2(n-3) + 2 * 2(n-4), \\ 0 &= B_4' B_3 = a_{14} B_1 B_3 + a_{24} B_2 B_3 + a_{34} B_3^2 + B_4 B_3 \\ &= a_{14} 2(n-3) + a_{24} 2 * 2(n-3) + a_{34} 2 * 3(n-3) + 2 * 3(n-4). \end{aligned}$$

From the first two equations, we calculate that $a_{14} = 0$. From the second and third, after substituting 0 for a_{14}, we calculate $a_{24} = 0$ and then $a_{34} = -(n-4)/(n-3)$. Similar patterns may be observed when the k^{th} equation is multiplied by vector-matrices $B_1, \ldots, B_{k-1}$. From these $k-1$ equations, we can see that all a_{ik} are equal to 0 except $a_{k-1,k} = -(n-k)/(n-k+1)$ which is the coefficient in formula (3).

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