On Distance-based Inconsistency Reduction Algorithms for Pairwise Comparisons

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Abstract

A complete proof of convergence of a certain class of reduction algorithms for distance-based inconsistency (defined in 1993) for pairwise comparisons is presented in this paper. Using pairwise comparisons is a powerful method for synthesizing measurements and subjective assessments. From the mathematical point of view, the pairwise comparisons method generates a matrix (say A) of ratio values (a_{ij}) of the ith entity compared with the jth entity according to a given criterion. Entities/criteria can be both quantitative or qualitative allowing this method to deal with complex decisions. However, subjective assessments often involve inconsistency, which is usually undesirable. The assessment can be refined via analysis of inconsistency, leading to reduction of the latter.

The proposed method of localizing the inconsistency may conceivably be of relevance for nonclassical logics (e.g., paraconsistent logic) and for uncertainty reasoning since it accommodates inconsistency by treating inconsistent data as still useful information.

Keywords: algorithm, convergence, pairwise comparisons, inconsistency

1 Basics of pairwise comparisons

In complex systems, making one comparison at a time is simpler than simultaneously assessing *all* entities, or components of a system, according to a given criterion. However, we need a method of synthesizing these partial assessments, especially when a decision is required. The pairwise comparisons method serves exactly this purpose, with the inconsistency analysis allowing us to localize the most questionable partial assessments and revise them if necessary.

From the mathematical point of view, the pairwise comparisons method creates a matrix (say A) of values (a_{ij}) of the ith entity compared with the jth entity:

$$A = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ \frac{1}{a_{12}} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \cdots & 1 \end{bmatrix}$$

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A scale $[\frac{1}{c}, c]$ is used for 'i to j' comparisons where $c > 1$ is a not-too-large real number (5 to 9 is used in most practical applications). It is usually assumed that all the values a_{ii} on the main diagonal are 1 (the case of 'i compared with i', that is with itself) and that matrix A is *reciprocal*: $a_{ij} = \frac{1}{a_{ji}}$ since 'i to j' is (or at least, is expected to be) the reciprocal of 'j to i'. (In other words, for $x, y \neq 0$, $\frac{x}{y} = \frac{1}{\frac{y}{x}}$.) However, in practice even the reciprocity condition is not always guaranteed. For example, in blind wine testing we may conclude that *wine* i is better than *wine* i if it is served in unmarked glasses. From an observer's point of view, there were two entities, but in reality there was only one wine. Still, this issue can be dealt with separately as shown, for example, in [12], and so in what follows we will work under the reciprocity hypothesis.

Matrix A is *consistent* if $a_{ij} \cdot a_{jk} = a_{ik}$ or equivalently: $a_{ij} \cdot a_{jk} \cdot a_{ki} = 1$) for every $i, j, k = 1, \ldots, N$. This is easily seen to be equivalent to $a_{ij} = \lambda_i / \lambda_j$ for *some* positive numbers $\lambda_1, \lambda_2, \ldots$ While every consistent matrix is reciprocal, the inverse fails in general (for all $n > 2$). Consistent matrices correspond to the ideal situation in which we know all exact values of all properties (or at least we think that we do). However, a realistic situation which is complex enough, nearly always involves inconsistency and we need to deal with it. In fact, when we are able to locate it, our comparisons can be reconsidered to reduce the inconsistency in the next round.

The pairwise comparisons method is still under intensive investigation (see, e.g., [4, 14, 16]). Advantages and deficiencies of various inconsistency indicators (such as those introduced in [17] and [11]) and the related inconsistency-reduction methods are being vigorously (sometimes too vigorously) debated, see for instance [1, 2, 15, 18]. Inspired by the publication of [4], efforts have been made here to address the issue of convergence of a certain class of reduction algorithms for the distance-based inconsistency in pairwise comparisons that were initially analyzed in [9]. Although familiarity with [4] is not required, it may be helpful for a better understanding of this brief presentation. Similarly, the reader may consult sections 2 and 5 of [9] – or alternatively the Appendix – to see how the results described here are used to reduce inconsistency. Appendix contains an example of how inconsistency analysis and reduction work in practice. Although not critical to understanding the mathematical proof of the convergence of distance-based inconsistency algorithms, it may be helpful to readers not familiar with the pairwise comparisons approach.

2 The distance-based inconsistency indicator

Since 1996, a *distance-based* adjective has been used by other researchers for the new inconsistency defined in 1993 in [11]. The distance-based adjective reflects the nature of the *inconsistency indicator*, which is defined, in essence, as a function of a distance from the nearest consistent *triad* in matrix A. Unlike the eigenvalue-based inconsistency introduced in [17], which is of a *global* nature, the distance-based inconsistency identifies the most inconsistent triad (or triads). It is the maximum over all triads $\{a_{ik}, a_{kj}, a_{ij}\}$ of elements of A (say, with all indices i, j, k distinct) of their inconsistency indicators, which in turn are defined as $ii = \min(|1 - \frac{a_{ij}}{a_{ik} a_{kj}}|, |1 - \frac{a_{ik} a_{kj}}{a_{ij}}|)$.

Comparing two entities often results in inaccuracy – that is, inexact knowledge – but it does not involve inconsistency. The minimal number of entities compared in pairs which may exhibit inconsistency in assessments is three. (In the general case, we need a cycle for inconsistency to manifest itself.) However, it is well-known and easy to show that (at least if the matrix A is complete, i.e., if all pairwise comparisons have been attempted) inconsistency is witnessed on some triad. In other words, the inconsistency indicator of A equals zero if and only if A is consistent (i.e., if indeed $a_{ij} = \lambda_i/\lambda_j$ for *some* positive numbers $\lambda_1, \lambda_2, \ldots$). This makes perfect sense: every inconsistency indicator should have value 0 for fully consistent systems, and *only* for fully consistent systems.

For the distance-based inconsistency, the highest value of the inconsistency indicator is for the following triads: $[\frac{1}{c}, c, \frac{1}{c}]$ and $[c, \frac{1}{c}, c]$. In both cases, it is $1 - \frac{1}{c^3}$. Assuming $c = 5$, $0 \le ii \le 0.992$. Certainly, inconsistency is undesirable in a system. On the other hand, although this may sound strange, it is not easy (we suspect, impossible) to construct a non-trivial *fully* inconsistent system: an "ideal" system where everything contradicts everything else. This question (or a family of questions, which we suggest only vaguely here) seems quite important as such impossibility would imply that *every* scenario of answers to pairwise comparison queries (even deliberately false) would necessarily create "apparent" consistencies.

In practical applications, a high value of the inconsistency indicator is a "red flag," or a sign of potential problems. A distance-based inconsistency reduction algorithm focuses, at each step, on an inconsistent triad and "corrects" it by replacing it with a consistent (or, more generally, less inconsistent) triad. It resembles "whac-a-mole," a popular arcade game. One difference is that instead of one mole, we have three array elements as explained above. After "hitting the mole" (which generally results in some other "moles" coming out), the next triad is selected according to some rule (which may be for example the greedy algorithm), and the process is repeated. Numerous practical implementations (e.g., a hazard rating system for abandoned mines in Northern Ontario) have shown that the inconsistency converges relatively fast. However, the need for rigorously *proving* the convergence (that is, showing that whacked moles *always* have the tendency of coming out less and less eagerly) was evident.

3 The new perspective

In 1996, an attempt to prove convergence of the above scheme was made in [9]. However, it has been discovered recently that the proof of Case C of Theorem 1 in [9] was incomplete. (The expression "messed up" may describe the situation more accurately.) Moreover, the cuts introduced into [9] during the editorial process made it rather difficult, even for an expert, to fix the proof, and to determine the correct quantitative estimates yielded by the approach.

Before an argument is provided to fill the gap and correct the error in the proof in [9], we feel obliged to share our findings related to the mathematical methods developed in the past. Theorem 1 of [9] turned out to be a special case of the convex feasibility problem which has a long history starting with [10] and more recently in [7]. Specifically, the Theorem (at least in its stated qualitative form) can be derived from Corollary 3.12 in [3], or from Theorem 1 (Global convergence) in [5]. The article [3] is one of the most comprehensive surveys on projection algorithms for solving convex feasibility problems. Received by the editors in 1993 (when [11] was published) and accepted for publication in 1995 (when [9] was submitted for publication), it is sixty pages long and may serve as a good source of information on this exciting theory applicable, for example, to the image recovery in computer tomography.

Thus, besides fixing the issues with the earlier paper [9], the present note establishes a connection between the extensive bodies of literature on inconsistency and pairwise comparisons on the one hand, and that on projection algorithms on the other hand. This connection went apparently unnoticed thus far. We also hope that our presentation will appeal to a diverse group of readers. It avoids unnecessary restrictions on algorithms (for example, it covers greedy algorithms which generally aren't cyclic - and there is no obvious reason why they should be intermittent, either; see [3] for definitions). It avoids generalizations to more abstract or infinite-dimensional settings, which are not necessary for the application and would be only confusing. It avoids appealing to compactness, and consequently leads to quantitative estimates on the rate of convergence. We emphasize that the quantitative estimates yielded by the present proof (see (3), (5) below) are *different* than those stated in [9].

To have an even clearer picture of the rate of convergence of particular algorithms, it would be helpful to determine, or at least reasonably estimate, the values of the parameter $\alpha = \mathrm{sep}(\mathcal{L})$ (that appears in the argument; see (1) and the paragraph that follows) in the specific case of the family of projections employed in the inconsistency reduction when n, the size of the system, is large. When $n=3$, we trivially have $\alpha=1$; when $n=4$, it is easy to check that $\alpha=\sqrt{2/3}$, and it appears that $\alpha=\sqrt{1/3}$ when $n=5$. While some inferences can be made from simulations, the case of larger n's remains somewhat mysterious. (It can not be resolved by brute force since, at least in the first analysis, the question is combinatorially intractable for large n.)

4 The proof of convergence

Below and in what follows p_U stands for the orthogonal projection onto a subspace U.

Theorem 1. *Let $\mathcal{L}$ be a non-empty finite family of the linear subspaces of $\mathbb{R}^N$. Let $W=\bigcap\mathcal{L}$ be the intersection of members of $\mathcal{L}$. Let $w:\mathbb{N}\to\mathcal{L}$ be a sequence such that for any $V\in\mathcal{L}$ the equality $w(n)=V$ holds for infinitely many $n\in\mathbb{N}$. Fix $x\in\mathbb{R}^N$ and define $x_0=x$ and*

$$x_n = p_{w(n)}(x_{n-1})$$

for $n\geq 1$. Then

$$\lim_n x_n = p_W(x).$$

Here is a complete proof of Theorem 1 along the lines of the argument from [9]. We will need some notation and an elementary lemma. If U, V are linear subspaces of $\mathbb{R}^N$, neither of which is contained in the other, we define their *separation* $\sigma(U,V)$ as the smallest non-zero principal angle between U and V. In other words, $\sigma(U,V)$ is the smallest angle between non-zero vectors $u\in U$ and $v\in V$, both of which are orthogonal to $U\cap V$. By compactness, $\sigma(U,V)$ is always strictly positive. If $U\subseteq V$ or $U\subseteq V$, we set $\sigma(U,V)=\pi/2$. (*Note*: This convention is different from, and more convenient than the one used in [9].) We have

Lemma 1. *Let U, V be linear subspaces of $\mathbb{R}^N$ and let $W=U\cap V$. If $u\in U$, then*

$$d(u,W)\leq \frac{d(u,V)}{\sin(\sigma(U,V))}.$$

PROOF. Denote $v = p_V(u)$ and $w = p_W(u)$, then $d(u, V) = |u - v|$ and $d(u, W) = |u - w|$. Since $p_W = p_W p_V$, it follows that $w = p_W p_V(u) = p_W(v)$. Hence $u - w$ and $v - w$ are both orthogonal to W. Accordingly, unless $v = w$ or $u = w$, in which case the assertion of the Lemma clearly holds), $u - w$ and $v - w$ can be can be used (in place of u, v) in the definition of $\sigma(U, V)$ and so the angle between them is at least $\sigma(U, V)$. The conclusion follows now from the fact that u, v and w form a right triangle with the right angle at v. ∎

Next, denote by $\mathcal{L}^\cap$ the family of all intersections of subfamilies of $\mathcal{L}$. The separation $\mathrm{sep}(\mathcal{L})$ of the family $\mathcal{L}$ is defined as

$$\mathrm{sep}(\mathcal{L}) = \min\{\sin\big(\sigma(V, V')\big) : V, V' \in \mathcal{L}^\cap\}. \tag{1}$$

As a minimum of a finite family of strictly positive numbers, $\mathrm{sep}(\mathcal{L})$ is strictly positive. (*Note*: For our proof it would have been enough to consider only pairs V, V' with $V \in \mathcal{L}$, $V' \in \mathcal{L}^\cap$.)

We now return to the proof of the Theorem 1. Let $y = p_W(x)$. Since $x_{n-1} - x_n$ is orthogonal to $x_n - y$, we have $|x_{n-1} - y|^2 = |x_{n-1} - x_n|^2 + |x_n - y|^2$ for every $n \in \mathbb{N}$. Thus, by induction, $|x_0 - y|^2 = \sum_{t=1}^{n} |x_{t-1} - x_t|^2 + |x_n - y|^2$ for every $n \in \mathbb{N}$. It follows that the sequence $(|x_n - y|)$ is non-increasing and that the infinite series

$$\sum_{t=1}^{\infty} |x_{t-1} - x_t|^2 \le |x_0 - y|^2 \tag{2}$$

is convergent.

We now claim that there exists a constant C, depending only on N and on $\mathcal{L}$, such that, for any m,

$$|x_m - y| \le C \sup_{n > m} |x_{n-1} - x_n| \tag{3}$$

Since (2) clearly implies that $\lim_{t\to\infty} |x_{t-1} - x_t| = 0$, the Theorem will readily follow. To prove the claim, fix m and set $\delta = \sup_{n>m} |x_{n-1} - x_n|$. Since every element of $\mathcal{L}$ appears in the sequence $w(t)$ infinitely many times, it follows that there exists $n \ge m$ such that $\{w(m), \ldots, w(n)\} = \mathcal{L}$ and hence $\bigcap_{t=m}^{n} w(t) = W$. We now define auxiliary sequences

$$W_t = \bigcap_{k=m}^{t} w(k) \qquad \text{and} \qquad y_t = p_{W_t}(x_m)$$

for $t = m, \ldots, n$. Since $W_n = W$ and, consequently, $y_n = y$ (in the notation of the statement of the Theorem), our objective is to majorize $x_m - y_n = d(x_m, W_n)$. The strategy will be to estimate, by induction with respect to t, the quantities $|x_m - y_t| = d(x_m, W_t)$.

We start by noting that $|x_m - y_m| = d(x_m, W_m) = d(x_m, w(m)) = 0$. Next, if $m < t \le n$, the following two cases need to be considered:

Case A. $W_t = W_{t-1}$ (or, equivalently, $W_{t-1} \subseteq w(t)$).

Then $y_t = y_{t-1}$ and $|x_m - y_t| = |x_m, y_{t-1}|$.

Case B. $W_t \subset W_{t-1}$ (strict inclusion; this implies, in particular, that $w(t) \notin \{w(s) : m \le s < t\}$).

The crucial point is that Case B may occur at most $M = \min\{N, \#\mathcal{L} - 1\}$ times, where # denotes cardinality. Case B combines Cases B and C from the original proof in [9].

By the triangle inequality

$$|x_m - y_t| \leq |x_m - y_{t-1}| + |y_{t-1} - y_t|$$

Now note that $y_t = p_{W_t}(y_m) = p_{W_t} p_{W_{t-1}}(y_m) = p_{W_t}(y_{t-1})$ and so, by Lemma 1 applied with $U = W_{t-1}$, $V = w(t)$ and $u = y_{t-1}$,

$$|y_{t-1} - y_t| = d(y_{t-1}, W_t) \leq \frac{d(y_{t-1}, w(t))}{\sin\sigma(W_{t-1}, w(t))}.$$

In turn,

$$d(y_{t-1}, w(t)) \leq |y_{t-1} - x_t| \leq |y_{t-1} - x_{t-1}| + |x_{t-1} - x_t| \leq |y_{t-1} - x_m| + \delta.$$

The last inequality follows from the definition of δ and from the fact that the sequence $(|y_{t-1} - x_s|)$, $s = m, m+1, \ldots, t-1$, is non-increasing (x_s being the image of x_{s-1} under the contraction p_{W_s} which also fixes y_{t-1}). Combining the last three estimates, we obtain

$$|x_m - y_t| \leq |x_m - y_{t-1}| + \frac{|y_{t-1} - x_m| + \delta}{\sin\sigma(W_{t-1}, w(t))} \leq (1 + \frac{1}{\alpha})|x_m - y_{t-1}| + \frac{\delta}{\alpha}, \quad (4)$$

where $\alpha = \mathrm{sep}(\mathcal{L})$. Since, as we noticed, Case B may occur at most M times, $|x_m - y_n|$ does not exceed the value obtained by M successive applications of the function $z \to \phi(z) = (1 + \frac{1}{\alpha})z + \frac{\delta}{\alpha}$, starting at $z = |x_m - y_m| = 0$. It is clear that that value is small when δ is small (the iteration is linear in δ). A more precise (routine but tedious) calculation shows that

$$|x_m - y| = |x_m - y_n| < \delta(1 + \alpha)(1 + 1/\alpha)^{M-1}, \quad (5)$$

which proves our claim (3). Here is a sketch. The sequence (a_k) whose Mth term we need to majorize is given by $a_0 = 0, a_{k+1} = \phi(a_k)$. Introduce an auxiliary sequence $b_k = (1 + 1/\alpha)^{-k} a_k$. Comparing b_{k+1} and b_k, we see that b_k is the kth partial sum of a geometric series, which can be explicitly computed and leads to the asserted estimate on $|x_m - y_n| \leq a_M$.

This completes the proof. Since each projection in the Theorem corresponds to correcting one of the triads (illustrated by a simple example in the Appendix and explained with more detail in [9]), it follows that, in terms of the previously used *whack-a-mole* metaphor, the moles have a declining tendency to show their heads. However, in practical situations, approximating an inconsistent pairwise comparisons matrix should take place only if the inconsistency indicator is "small enough." There is no precise formula for deciding the threshold value. In practice, some kind of heuristic is used. However, if we follow the greedy algorithm (that is, locate the most inconsistent triad and reduce its inconsistency) or any other reasonable strategy, the above theorem guarantees the required convergence to a fully consistent matrix. We do not, however, (or at least should not) blindly reduce the inconsistency to 0. Solving the dilemma "how" is left to the user (a human expert). It is assumed, however, that he/she is able to reduce the inconsistency at each step since this is based on checking the equality $a_{ij} = a_{ik} \cdot a_{kj}$.

5 Concluding remarks, reflections, and future plans

Our own subjective assessments are often inconsistent. They can be refined by the inconsistency analysis. Although such analysis can help nearly always, it is rarely done. In fact, there are very few method for processing subjectivity. This presentation has also stressed the importance of inconsistency analysis in the pairwise comparisons process.

In the past, this method was used by the first named author to develop AMIS (Abandoned Mines Hazard Rating System) for the government of Ontario (The Ministry of Northern Ontario and Mines). The system ranked an abandoned mine, located in Northern Ontario, as one of the most dangerous from a public safety point of view. Its eventual collapse convinced the government that its research funding was well spent.

Monte Carlo studies have shown that approximations of highly inconsistent pairwise comparisons matrices yields high errors. Finding consistent approximations of such matrices makes little practical sense. From the standard mathematical logic, we know that *only* falsehood can generate both truth or falsehood. Fuzzy logic (at least in its naive form) would suggest that *a little bit of falsehood should still largely lead to truth.* However, the old adage that *one bad apple spoils the barrel* seems to be more applicable here: even a little bit of falsehood may contribute to massive errors and misjudgments. An approximation of a pairwise comparisons matrix is meaningful only if the initial inconsistency is acceptable (that is, located, brought under control and/or reduced to a certain predefined minimum; in our analogy, *always remove overripe fruit promptly if it is possible to find it*).

Even though the specific procedures analyzed in [9] are not fully satisfactory (since, if applied blindly, they lead to the logarithmic least square approximation and the possible propagation of falsehoods), they constitute a necessary step toward a better understanding of the inconsistency concept. This paper shows that inconsistency analysis may be used for improving our assessments before approximations are computed. Until a better solution for handling inconsistency is found, we need to assume that a highly inconsistent pairwise comparisons matrix is meaningless, since this assumption is safer than the use of risky numbers derived from a false antecedent.

By asserting an implication, one usually claims that the antecedent cannot be true if the consequent is false. However, classical logic does not address the dilemma of "a little bit false" and knowing at least where such little falsehood takes place is essential.

The new results and applications of pairwise comparisons show the importance of the consistency-driven approach. In particular, the application of this approach to knowledge-based systems (e.g., presented in [13]) has turned out to be fruitful. The inconsistency concept still remains enigmatic and more research needs to be done. In particular, inconsistency in a general system needs to be defined and this study is a step forward. The idea of improving inaccuracy by controlling inconsistency cannot be wrong. Knowing what we do not know is essential to managing the knowledge and improving it. On the other hand, it is hard to refine our knowledge if we choose not to know what we know or even should know.

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Appendix: Inconsistency analysis illustrated by a simple example

Making comparative assessments especially of some intangible properties (such as public safety or reliability) results not only in imprecise knowledge, but also involves inconsistency in our assessments. This Appendix illustrates how the knowledge can be refined by controlling inconsistencies in these assessments.

Needless to say, the solution of matrices (that is, finding the vector of weights) strongly depends on the inconsistency. For inconsistent matrices, it is only an approximation and frankly, no one really knows how imprecise it is in terms of how much the matrix, reconstructed from the vector of weights differs, from the given matrix. Hence analyzing and minimizing inconsistency is an essential step in pairwise comparisons. Even though, we can establish the distance (e.g., Euclidean) between the reconstructed matrix and the given matrix, its interpretation is only along the good or bad. More empirical research is needed to establish some good practices or heuristics for the acceptable threshold of inaccuracy.

The main challenge to the pairwise comparisons method comes from a lack of consistency in the pairwise comparisons matrices that occurs in most practical applications. In fact, ‘‘in almost all practical applications’’ is a better approximation that ‘‘in most practical applications’’.

Consider the following example. Suppose, we estimate ratios A/B as 1, B/C as 4, and A/C as 5. Evidently something does not ‘‘add up’’ because $\frac{A}{B} \cdot \frac{B}{C} = 1 \cdot 2 = 2$, which obviously is not equal to 5 (that is, the value of $\frac{A}{C}$). With an inconsistency index of 0.60 the above triad is ‘‘boxed’’ as the most inconsistent in the entire matrix.

	A	**B**	**C**	**D**
A	1	$\boxed{1}$	$\boxed{5}$	4
B	1	1	$\boxed{2}$	$2\frac{1}{2}$
C	$\frac{1}{5}$	$\frac{1}{2}$	1	$\frac{1}{2}$
D	$\frac{1}{4}$	$\frac{2}{5}$	2	1

(6)

Changing the value 1 in the above triad to 2.5 makes this triad fully consistent since $2.5 \cdot 2 = 5$. Unfortunately, this is not the end of our problems since there is another triad $[2, 2\frac{1}{2}, \frac{1}{2}]$ that is inconsistent and ‘‘boxed’’ below:

	A	**B**	**C**	**D**
A	1	$2\frac{1}{2}$	5	4
B	$\frac{2}{5}$	1	$\boxed{2}$	$\boxed{2\frac{1}{2}}$
C	$\frac{1}{5}$	$\frac{1}{2}$	1	$\boxed{\frac{1}{2}}$
D	$\frac{1}{4}$	$\frac{2}{5}$	2	1

(7)

Assume that we have good reason (coming from the knowledge domain; not from mathematics), to change the value of $2\frac{1}{2}$ to 1. It is an arbitrary decision since 2 could have been changed to 5 or $\frac{1}{2}$ to $1\frac{1}{4}$ also making this triad consistent. Only the domain knowledge can determine the change of the value (or values) in a triad. However, changing 2 may not be wise since it belongs to a consistent triad altered in the previous step. In our case, the only reason why we have chosen to change $2\frac{1}{2}$ to 1 was to illustrate how the inconsistency procedure works and the reader may be disappointed to find that there is yet another triad "boxed" below which is inconsistent:

$$\begin{array}{ccccc} & \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} \\ \mathbf{A} & 1 & \boxed{2\frac{1}{2}} & 5 & \boxed{4} \\ \mathbf{B} & \frac{2}{5} & 1 & 2 & \boxed{1} \\ \mathbf{C} & \frac{1}{5} & \frac{1}{2} & 1 & \frac{1}{2} \\ \mathbf{D} & \frac{1}{4} & 1 & 2 & 1 \end{array} \tag{8}$$

Finally, we change 4 to $2\frac{1}{2}$ making the entire table fully consistent.

$$\begin{array}{ccccc} & \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} \\ \mathbf{A} & 1 & 2\frac{1}{2} & 5 & 2.5 \\ \mathbf{B} & \frac{2}{5} & 1 & 2 & 1 \\ \mathbf{C} & \frac{1}{5} & \frac{1}{2} & 1 & \frac{1}{2} \\ \mathbf{D} & \frac{2}{5} & 1 & 2 & 1 \end{array} \tag{9}$$

In practice, inconsistent assessments are unavoidable when at least three factors are independently compared against each other. The corrections to real data are done on the basis of professional experience, the case-based knowledge, and by the careful examination of all criteria involved (not necessarily in the current triad).

An acceptable threshold of inconsistency, for most practical applications, turns out to be $\frac{1}{3}$. This is so because one value in a triad is not more than two grades off the scale from the remaining two values. This heuristic was introduced in [11] and it seems more mathematically sound than 10% proposed in [17].

There is no need to continue decreasing the inconsistency indefinitely to zero, as only a high value of it is harmful. In fact, a zero or a small inconsistency value may indicate that artificial data were entered hastily without reconsideration of former assessments, which is an unacceptable practice.

For the improved matrix, the normalized vector of weights is:

$$w=[0.5,0.2,0.1,0.2]$$

Vector w identical for both the geometric means method, and the eigenvector method, since the eigenvector of a consistent pairwise comparisons matrix is always equal to the geometric means. For the original input matrix, which is inconsistent, the solutions are,
for the eigenvector method:

$$w=[0.441,0.317,0.101,0.140]$$

and for geometric means method (computed as $w_i=\sqrt[n]{\prod_{j=1}^{n} a_{ij}}$):

$$w=[0.445,0.315,0.100,0.141]$$

The difference between both solutions is negligible. However, both solutions for the inconsistent matrix vary significantly from the solution for the consistent matrix.

It is important to note the difference between inaccuracy and inconsistency. For example, in a triad $[2,5,3]$, a rash approach may lead us to believe that A/C should indeed be 6 since it is $2\cdot 3$, but we do not have any reason to reject the estimation of B/C as 2.5 or A/B as 5/3.

This is what inconsistency is about. It is not inaccuracy, but when used wisely, it may help to decrease inaccuracy.

The reader will notice that while the three-step inconsistency-reduction procedure performed above does not offend the common sense, it is rather *ad hoc*, hence not fully satisfactory. This remark applies both to the choices of triads to be corrected, and to the choices of the particular members of each such triad that is being modified. The algorithm analyzed in [9] (and, by extension, the present note) is more canonical with respect to the second point. In general, it replaces the triad $\{a_{ik}, a_{kj}, a_{ij}\}$ by $\{a_{ik}/r, a_{kj}/r, ra_{ij}\}$, where $r=\sqrt[3]{a_{ik}, a_{kj}/a_{ij}}$. This corresponds to subtracting from the matrix $(\log a_{uv})$ its orthogonal projection onto the direction of the skew-symmetric matrix $B=(b_{uv})$ defined by the requirement that $a_{ik}=1=a_{kj}$, $a_{ij}=-1$ and that all other super-diagonal entries are 0; the corresponding subspace in the context of Theorem is $U=\{X : X$ is an $n\times n$ skew-symmetric matrix such that $\mathrm{tr}BX=0\}$. In particular, for the first triad $[1,2,5]$ considered above, we have $r=2/5$ and the corrected triad is $[\sqrt[3]{5/2}, 2\sqrt[3]{5/2}], 5\sqrt[3]{2/5}]\approx[1.36, 2.71, 3.68]$.