

Pairwise Comparisons Simplified

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Abstract

This study examines the notion of generators of a pairwise comparisons matrix. Such approach decreases the number of pairwise comparisons from $n \cdot (n - 1)$ to $n - 1$. An algorithm of reconstructing of the PC matrix from its set of generators is presented.

Keywords: *pairwise comparisons, generator, graph, tree, algorithm*

1 Introduction

In [17], Thurstone proposed “The Law of Comparative Judgments” for pairwise comparisons (for short, PC). However, the first use of pairwise comparisons is in [4]). Even earlier, Condorcet used it in [3], but in a more simplified way for voting (win or loss).

The approach, based on the additive value model is presented in [1]. Generalization of the pairwise comparisons method was examined in [15]. In [5], the PC matrix approximation problem was examined.

In this study, we examine the possibility of reconstructing the entire $n \times n$ PC matrix from only $n - 1$ given entries placed in strategic locations. We

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call them generators. Before we progress, some terminologies of pairwise comparisons must be revisited in the next section, since PC theory is still not as popular as other mathematical theories. However, the next section is definitely not for PC method experts.

2 Pairwise comparisons basics

We define an $n \times n$ *pairwise comparison matrix* simply as a square matrix $M = [m_{ij}]$ such that $m_{ij} > 0$ for every $i, j = 1, \dots, n$. A pairwise comparison matrix M is called *reciprocal* if $m_{ij} = \frac{1}{m_{ji}}$ for every $i, j = 1, \dots, n$ (then automatically $m_{ii} = 1$ for every $i = 1, \dots, n$). Let us assume that:

$$M = \begin{bmatrix} 1 & m_{12} & \cdots & m_{1n} \\ \frac{1}{m_{12}} & 1 & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m_{1n}} & \frac{1}{m_{2n}} & \cdots & 1 \end{bmatrix}$$

where m_{ij} expresses a relative quantity, intensity, or preference of entity (or stimuli) E_i over E_j . A more compact and elegant specification of PC matrix is given in [14] by Kulakowski.

A pairwise comparison matrix M is called *consistent* (or *transitive*) if:

$$m_{ij} \cdot m_{jk} = m_{ik}$$

for every $i, j, k = 1, 2, \dots, n$.

We will refer to it as a “consistency condition”. Consistent PC matrices correspond to the situation with the exact values $\mu(E_1), \dots, \mu(E_n)$ for all the entities. In such case, the quotients $m_{ij} = \mu(E_i)/\mu(E_j)$ then form a consistent PC matrix. The vector $s = [\mu(E_1), \dots, \mu(E_n)]$ is unique up to a multiplicative constant. While every consistent matrix is reciprocal, the converse is generally false. If the consistency condition does not hold, the matrix is *inconsistent* (or *intransitive*).

Axiomatization of inconsistency indicators for pairwise comparisons has been recently proposed in [13].

The challenge for the pairwise comparisons method comes from the lack of consistency of the pairwise comparisons matrices, which arises in practice (while as a rule, all the pairwise comparisons matrices are reciprocal). Given

an $N \times N$ matrix M , which is not consistent, the theory attempts to provide a consistent $n \times n$ matrix M , which differs from matrix M “as little as possible”.

It is worth to noting that the matrix: $M = [v_i/v_j]$ is consistent for all (even random) positive values v_i . It is an important observation since it implies that a problem of approximation is really a problem of a norm selection and the distance minimization. For the Euclidean norm, the vector of geometric means (equal to the principal eigenvector for the transitive matrix) is the one which generates it. Needless to say that only optimization methods can approximate the given matrix for the assumed norm (e.g., LSM for the Euclidean distance, as recently proposed in [7]). Such type of matrices are examined in [16] as “error-free” matrices.

It is unfortunate that the singular form “comparison” is sometimes used considering that a minimum of three comparisons are needed for the method to have a practical meaning. Comparing two entities (stimuli or properties) in pairs is irreducible, since having one entity compared with itself gives trivially 1. Comparing only two entities (2×2 PC matrix) does not involve inconsistency. Entities and/or their properties are often called stimuli in the PC research but are rarely used in applications.

3 The generators of pairwise comparisons matrix

For a given PC matrix $A \in M_{n \times n}(\mathbb{R})$ consider the set $C_n := \{a_{ij} : i < j\}$. Note that to reconstruct the whole matrix it is enough to know the elements of C_n , as $a_{ii} = 1$ for each $i \in \{1, \dots, n\}$ and $a_{ji} = \frac{1}{a_{ij}}$ for $i < j$.

Let us call each such set sufficient to reconstruct the matrix A its *set of generators*.

The set C_n has $\frac{n^2-n}{2}$ elements. However, consistency is a much stronger condition. So, it is obvious that we can reduce this input set to calculate the rest of elements. It is a natural question to ask which minimal subsets of C_n generate A .

Remark 3.1. *If $B \subset B' \subset C_n$ and B generates A , then B' does as well.*

Theorem 3.2. *There is no $(n - 2)$ -set of generators of A .*

Proof. For $n = 3$ the statement is obvious, as in any matrix:

$$\begin{bmatrix} 1 & a & c \\ \frac{1}{a} & 1 & b \\ \frac{1}{c} & \frac{1}{b} & 1 \end{bmatrix}.$$

if we only know one of the values a , b or c , we cannot clearly calculate the other two satisfying $c = ab$.

To continue the induction, let us assume that the assertion holds for each matrix $M \in M_{n \times n}(\mathbb{R})$. Now consider the matrix:

$$A_{n+1} = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ \frac{1}{a_{12}} & 1 & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \cdots & 1 & a_{n,n+1} \\ \frac{1}{a_{1,n+1}} & \frac{1}{a_{2,n+1}} & \cdots & \frac{1}{a_{n,n+1}} & 1 \end{bmatrix}$$

Notice that in order to calculate the elements of the last column, we need to know at least one of them. On the other hand, if we know p of them, we can calculate only $p - 1$ of the elements a_{ij} for $1 \leq i < j \leq n$. Let us assume there is a $(n - 1)$ -set B of generators of A_{n+1} . We define a new set $B' := B \cup L \setminus R$, where R denotes the set of elements of B from the last column, and L denotes the elements from the previous columns which can be calculated from the elements of R . Now B' is a $(n - 2)$ -set of generators of the matrix A_n resulting from A_{n+1} by removing the last row and column.

$A_n \in M_{n \times n}(\mathbb{R})$, which contradicts the inductive assumption. □

Remark 3.3. *Given input values $a_{i,i+1}$ (for $i = 1, \dots, n - 1$) located above the main diagonal, from the consistency condition we can reconstruct the entire matrix*

$$A = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ \frac{1}{a_{12}} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \cdots & 1 \end{bmatrix}$$

using the formula

$$a_{ij} = \prod_{k=i}^{j-1} a_{k,k+1} \tag{1}$$

for $j > i$.

It is worth mentioning that the above $n - 1$ values are not the only values generating the entire matrix. The following theory is provided for finding the other minimal sets of generators.

Remark 3.4. *There is a mutual relevance between each set $B \subset C_n$ and an undirected graph G_B with n vertices:*

$$a_{ij} \in B \Leftrightarrow \text{there is an edge } i - j \text{ in } G_B.$$

Lemma 3.5. *If we know k edges of any subtree of G_B , then we are able to compute $\binom{k+1}{2}$ of elements of C_n .*

Proof. For $k = 1$ the statement is obvious, as $\binom{1+1}{2} = 1$.

To proceed with induction, let us assume that the statement is true for k . Take a tree D with $k + 2$ vertices $x_1, \dots, x_{k+2} \in V(D)$ and $k + 1$ edges. Remove any leaf $x_l \in V(D)$ together with the edge $x_m - x_l$, joining the leaf with the tree. We get a new tree D' with k edges. From the inductive assumption, we are able to calculate $\binom{k+1}{2}$ of elements of C_n .

Now, when we give the removed edge back, we notice that for every vertex $x_j \in V(D) \setminus \{l\}$ there is a path $x_j - x_{p_1} - \dots - x_{p_s} - x_l$ joining x_j with x_l and we can compute $a_{jl} = a_{jp_1} \cdot a_{p_1 p_2} \cdot \dots \cdot a_{p_s l}$. These are $k + 1$ new elements and altogether we know $\binom{k+1}{2} + k + 1 = \binom{k+2}{2}$ of elements of C_n . \square

Now we can formulate the necessary and sufficient condition for minimal sets of generators.

Theorem 3.6. *Let us assume $B \subset C_n$ is a $(n - 1)$ -set. Then*

$$B \text{ generates matrix } A \Leftrightarrow G(B) \text{ is a tree.}$$

Proof. Take a set B of generators of matrix A . Let us assume $G(B)$ is not a tree. If so, it must contain a cycle $x_{k_1} - x_{k_2} - \dots - x_{k_s} - x_{k_1}$. When we remove the edge $x_{k_1} - x_{k_s}$ the relevant $n - 2$ -set still generates A , as $a_{k_1, k_s} = a_{k_1, k_2} \cdot \dots \cdot a_{k_{s-1}, k_s}$. Thus we get a contradiction with Theorem 3.2.

The reverse implication follows straight from Lemma 3.5 for $k = n - 1$. \square

Corollary 3.7. *There are n^{n-2} minimal sets of generators of A .*

Proof. This is an immediate consequence of the Cayley's formula for the number of trees on n vertices. \square

Although there are many combinations of $n - 1$ values generating the entire PC matrix, the values $m_{i,i+1}$ are the most important of all combinations since they express this sequence:

$$E_1/E_2, E_2/E_3, \dots, E_{n-2}/E_{n-1}, E_{n-1}/E_n \quad (2)$$

Let us call the above $n - 1$ values as *principal generators* (PGs). For matrix A_n , the principal generators are located above the main diagonal and are as follows:

$$a_{1,2}, a_{2,3}, \dots, a_{n-1,n}$$

Let us invent our own handicapping. Handicapping, in sports and games, is the practice of assigning advantage through scoring compensation or other advantage given to different contestants to equalize the chances of winning. The same term also applies to the various methods for computing advantages.

Entities 1 and n occurs in (2) only once. Any other entity, 2 to $n - 1$, occurs twice. Since the highest frequency is 2, it is fair to add 1 to E_1 and E_n to compensate for only one occurrence and count the maximum number of occurrences of an entity to compensate other entities. We also define the total handicapping as the total of all compensations.

By the *frequency* $f(i, B)$ of an entity $i \in \{1, \dots, n\}$ in a set $B \subset C_n$ we understand the cardinality of set

$$B_i := \{a_{jk} \in B : j = i \vee k = i\}.$$

Let $O(B)$ denote the set $\{i : B_i \neq \emptyset\}$. We define the *total handicapping* of the set $B \subset C_n$ as

$$h(B) := \sum_{i \in O(B)} \left(\max_{j \in O(B)} f(j, B) - f(i, B) \right).$$

For example, for generators assumed to be the first row, the total handicapping would be $(n - 2)(n - 1)$, since E_1 occurs $n - 1$ times and the rest entities only once, so they need to be handicapped by $n - 2$.

Remark 3.8. *The frequency of an entity i in the set B is equal to the degree of vertex i in graph G_B :*

$$f(i, B) = \deg_{G_B}(i).$$

Remark 3.9. *$O(B)$ is the set of vertices of G_B with a positive degree.*

Remark 3.10. *$h(B)$ counts the sum of differences between the maximal degree of a vertex in G_B and the degrees of the rest of vertices.*

Corollary 3.11. *$h(B) = 0 \Rightarrow$ graph G_B is regular (all its vertices have the same degree).*

Theorem 3.12. *If $n > 1$ and B is a $(n - 1)$ -set of generators of A , then*

1. $h(B) = 0 \Rightarrow n = 2$
2. $h(B) \neq 1$
3. $h(B) = 2 \Rightarrow G_B$ is a path connecting all vertices.

Proof. From Theorem 3.6, it follows that $G(B)$ is a tree. Thus, it has n vertices and $n - 1$ edges. Notice that to obtain the number of edges one needs to sum up the degrees of vertices and divide by two (each edge is counted twice).

If $h(B) = 0$, then $G(B)$ has n vertices of the same degree k . Hence,

$$n - 1 = \frac{nk}{2} \Rightarrow 2n - 2 = nk \Rightarrow n(2 - k) = 2 \Rightarrow n = 2.$$

If $h(B) = 1$, then $G(B)$ has $n - 1$ vertices of the same degree k and one of degree $k - 1$. Hence,

$$\begin{aligned} n - 1 &= \frac{(n - 1)k}{2} + \frac{k - 1}{2} = \frac{nk - 1}{2} \Rightarrow 2n - 2 = nk - 1 \Rightarrow \\ &\Rightarrow (2 - k)n = 1 \Rightarrow n = 1, \end{aligned}$$

which is a contradiction.

If $h(B) = 2$, then there are two cases. The first one with $n - 1$ vertices of degree k and one vertex of degree $k - 2$. Hence,

$$n - 1 = \frac{(n - 1)k}{2} + \frac{k - 2}{2} = \frac{nk - 2}{2} \Rightarrow 2n - 2 = nk - 2 \Rightarrow k = 2,$$

and it follows that there is one vertex of degree 0, so graph G_B is disconnected, which is a contradiction.

For a case with $n - 2$ vertices of degree k and two vertices of degree $k - 1$, we have:

$$n - 1 = \frac{(n - 2)k}{2} + \frac{2}{k - 1} \cdot 2 = \frac{nk - 2}{2} \Rightarrow 2n - 2 = nk - 2 \Rightarrow k = 2,$$

and it follows that G_B is a path connecting all vertices. □

Corollary 3.13. *For $n \geq 2$ there are $\frac{n!}{2}$ $(n - 1)$ -sets of generators of A with the minimal total handicapping.*

Proof. For $n = 2$ the statement is obvious. For $n \geq 3$ from Theorem 3.12, we know that it suffices to count all the paths connecting all vertices. Each such path is of the form $p(1) - p(2) - \dots - p(n)$, where p is a permutation of the set $\{1, \dots, n\}$. However, from $p(1) - p(2) - \dots - p(n)$ and $p(n) - p(n - 1) - \dots - p(1)$, we obtain the same tree hence the number of permutations must be divided by two. □

Example 3.14. Consider $n = 4$ and the matrix

$$A = \begin{bmatrix} 1 & a & b & c \\ \frac{1}{a} & 1 & d & e \\ \frac{1}{b} & \frac{1}{d} & 1 & f \\ \frac{1}{c} & \frac{1}{e} & \frac{1}{f} & 1 \end{bmatrix}.$$

The set $C_4 = \{a, b, c, d, e, f\}$ consists of $\frac{4^2 - 4}{2} = 6$ elements, so it has

$\binom{6}{3} = 20$ 3-subsets listed in Tab. 3.14. From Remark 3.4, we know that they are related to some graphs. They are listed in Fig. 1 and their nodes are labeled in Fig. 2.

The last four subsets are triads, so they are not generators of A . The graphs related to them are cycles.

According to Corollary 3.7, there are $4^{4-2} = 16$ minimal sets of generators of A and they are the first 16. The graphs related to them are obviously trees. From Corollary 3.13, we conclude that we have $\frac{4!}{2} = 12$ sets of generators of A minimizing the total handicapping and they are the first 12. The related graphs are paths.

It is worth noticing that the first subset is the set of PGs.

$\{a, d, f\}$	$\{a, c, d\}$	$\{a, c, f\}$	$\{c, d, f\}$
$\{a, e, f\}$	$\{a, b, f\}$	$\{b, c, d\}$	$\{c, d, e\}$
$\{a, b, e\}$	$\{b, c, e\}$	$\{b, e, f\}$	$\{b, d, e\}$
$\{a, b, c\}$	$\{a, d, e\}$	$\{b, d, f\}$	$\{c, e, f\}$
$\{a, c, e\}$	$\{a, b, d\}$	$\{b, c, f\}$	$\{d, e, f\}$

Table 1: 3-subsets for $n = 4$

The strategic importance of PGs, based on the total handicapping, places them on the diagonal above the main diagonal. It is important to notice that PGs are always there and it is up to us to decide whether or not we use them for the reconstructing the entire consistent PC matrix (with the possible increased error) or enter the remaining entires into PC matrix with the possibility of inconsistency in triads. As pointed out in [11], nothing is carved in stone and even reciprocity can be relaxed for the blind wine and other types of tasting. Reciprocity may also not be guaranteed when data are collected over the Internet from different sources or individuals. For PGs sufficiently accurate, the PC matrix reconstruction is a vital possibility.

4 An algorithm for the reconstruction of a PC matrix from generators

Assuming that we know the $(n - 1)$ -set $B \subset C_n \subset A$, we can reconstruct the entire matrix A using an algorithms described below.

For given PGs a_{11}, \dots, a_{nn} , we can compute each a_{ij} for $i < j$ from Remark 3.3. On the other hand, if we know a $(n - 1)$ -set B of different values of matrix A we can apply the log function to the formula (1) and we substitute $x_k := \log a_{k,k+1}$ then we get the system of linear equations

$$\log a_{ij} = \sum_{k=i}^{j-1} x_k, \quad a_{ij} \in B$$

If B generates A , then this system has a unique solution and we may calculate $a_{k,k+1} = 10^{x_k}$.

This leads us to the following algorithm:

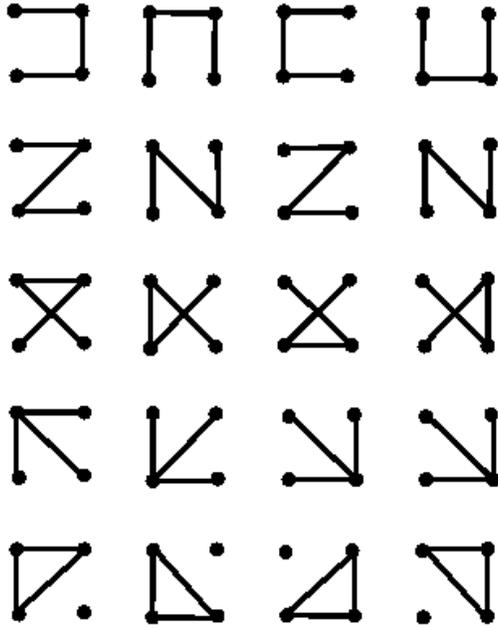


Figure 1: All possible combinations of 20 three generators

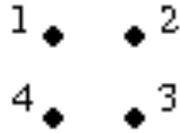


Figure 2: Nodes for all generators in Fig. 1

INPUT:

- $M \in \mathcal{M}(n - 1, 3)$ representing the set B of generators of A .
- $(M[i, 1], M[i, 2])$ correspond to the coordinates of the i -th element of B .
- $M[i, 3]$ corresponds to the value of the i -th element of B .

OUTPUT:

The consistent PC matrix $A \in \mathcal{M}(n, n)$ reconstructed from A .

ALGORITHM:

1. Construct graph G_B :
 $G_B := \emptyset$;
 for $i := 1$ to $n - 1$ $G_B := G_B \cup \{M[i, 1] - M[i, 2]\}$;
2. Check if G_B is a tree (using DFS algorithm):
 if not then write (' B does not generate A ') and exit;
3. Solve a linear system

$$\sum_{k=M[i,1]}^{M[i,2]-1} x_k = \log(M[i, 3]), \quad i = 1, \dots, n - 1$$

4. Calculate elements of A :
 for $k := 1$ to n $A[k, k] := 1$;
 for $k := 1$ to $n - 1$ $A[k, k + 1] := 10^{x_k}$;
 for $k := 1$ to $n - 1$
 for $l := k + 2$ to n $A[k, l] := \prod_{m=k}^{l-1} A[m, m + 1]$
 for $k := 1$ to $n - 1$
 for $l := k + 1$ to n $A[l, k] := \frac{1}{A[k, l]}$

Example 4.1. Let us consider the PC matrix A , from the example 3.14, and assume that we know the values from the set $B = \{a, b, e\}$. Thus, the input matrix is:

$$M = \begin{bmatrix} 1 & 2 & a \\ 1 & 3 & b \\ 2 & 4 & e \end{bmatrix}.$$

Apply the algorithm:

1. We obtain the graph G_B with 4 vertices

$$1, 2, 3, 4$$

and 3 edges

$$1 - 2, 1 - 3, 2 - 4.$$

2. We assure that G_B is a tree and continue.
3. We solve the linear system

$$\begin{cases} x_1 = \log a \\ x_1 + x_2 = \log b \\ x_2 + x_3 = \log e \end{cases}$$

and we get

$$\begin{cases} x_1 = \log a \\ x_2 = \log b - \log a = \log \frac{b}{a} \\ x_3 = \log e - \log b + \log a = \log \frac{ae}{b} \end{cases}$$

4. We calculate the missing elements of the output PC matrix A :

$$A = \begin{bmatrix} 1 & a & b & ae \\ \frac{1}{a} & 1 & \frac{b}{a} & e \\ \frac{1}{b} & \frac{a}{b} & 1 & \frac{ae}{b} \\ \frac{1}{ae} & \frac{1}{e} & \frac{b}{ae} & 1 \end{bmatrix}.$$

Remark 4.2. *The complexity of the algorithm is $O(n^3)$.*

Proof. Steps 1 and 2 take $O(n)$ operations. The complexity of step 3 is $O(n^2)$ and of step 4 is $O(n^3)$. \square

Notice that the assumption that the number p of input entries is equal to $n - 1$ may not be satisfied. We may have less or more data and still use the slightly modified algorithm. We only change steps **1** and **2**:

1. Construct graph G_B :
 $G_B := \emptyset$;
for $i := 1$ to p $G_B := G_B \cup \{M[i, 1] - M[i, 2]\}$;
2. Replace G_B by its spanning tree received from DFS algorithm.
if G_B has less than $n - 1$ edges then write (' B does not generate A ')
and exit;

5 Conclusions

In this study, we have demonstrated that we are able to use less than all $n \cdot (n - 1)/2$ pairwise comparisons to reconstruct a consistent matrix. In fact, the minimum of $n - 1$ generators is sufficient. This may be very useful, provided that they represent more accurate values than the rest of the PC matrix. Another possible application of the suggested algorithm is a case of the missing data in the PC matrix. The entire PC can be reconstructed from the principal generators located above the main diagonal. Together with the preliminary sorting of entities, it is quick and easy way of getting weights.

The problem is that the accumulated errors grow fast when we “walk away” from the main diagonal towards the upper right corner. In fact, for $n = 7$ and 20% error in the principal generators, nearly 200% is accumulated in the upper right corner.

The proposed algorithm for reconstructing the entire matrix from the principal generators is easy to implement, even in Gnumeric (or in MS Excel). In addition, the reconstructed values (accurate or not) are consistent.

In this study, we only consider the multiplicative variant of PC, which is based on “*how many times?*”, while the additive version of pairwise comparisons (“*by how much?*”) was recently analyzed in [18]. The additive PC method utilizes a different type of inconsistency (not addressed here). Certainly more research is needed.

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