On Axiomatization of Inconsistency Indicators for Pairwise Comparisons

W.W. Koczkodaj *   R. Szwarc †

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Abstract

This study examines the notion of inconsistency in pairwise comparisons for providing an axiomatization for it. It also proposes two inconsistency indicators for pairwise comparisons. The primary motivation for the inconsistency reduction is expressed by a computer industry concept "garbage in, garbage out". The quality of the output depends on the quality of the input.

Keywords: *pairwise comparisons, inconsistency axiomatization*

1 Introduction

The method of pairwise comparisons (PC method here) is attributed to Fechner (see [5]) as a formal scientific method although it was first mentioned by Condorcet in [4] who only used it in its primitive form: win/loss. However, Thurstone (see [19]) proposed what is known as "The Law of Comparative Judgments" in 1927. In 1977, Saaty proposed his AHP method based on modified pairwise comparisons with a hierarchy structure in [16]. In this study, however, the hierarchy is not considered. It is also worth to note that

*Computer Science, Laurentian University,Sudbury, Ontario P3E 2C6, Canada wkoczkodaj@cs.laurentian.ca

†Institute of Mathematics, University of Wroclaw, Wroclaw, Poland, szwarc2@gmail.com

in this study, we consider only the multiplicative PC which is based on *"how many times?"*, while the additive version of pairwise comparisons ("by how much...")was recently analyzed in [21]. It has a different type of inconsistency (not addressed here).

Saaty's seminal study [16] had a profound impact on the pairwise comparisons research. However, his AHP should not be equalized with pairwise comparisons, despite using them. The restrictions assumed by Saaty (e.g., fixed scale: 1 to 9) probably serves its proponent well for whatever purpose he has designed it. However, it is a subset of the pairwise comparisons for which no particular scale is assumed. A proof was provided in [6] that a small scale (1 to 3) has desired mathematical properties for the use in pairwise comparisons.

Regretfully, pairwise comparisons theory is not as popular as in mathematics, for example, partial differential equations, hence basic concepts need to be presented in the next section but it is not PC method experts.

2 Pairwise comparisons basics

We define an $N \times N$ pairwise comparison matrix simply as a square matrix $M = [m_{ij}]$ such that $m_{ij} > 0$ for every $i, j = 1, \ldots, n$. A pairwise comparison matrix M is called *reciprocal* if $m_{ij} = \frac{1}{m_{ji}}$ for every $i, j = 1, \ldots, n$ (then automatically $m_{ii} = 1$ for every $i = 1, \ldots, n$). Let us assume that:

$$M = \begin{bmatrix} 1 & m_{12} & \cdots & m_{1n} \\ \frac{1}{m_{12}} & 1 & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{m_{1n}} & \frac{1}{m_{2n}} & \cdots & 1 \end{bmatrix}$$

where m_{ij} expresses a relative preference of entity (or stimuli) s_i over s_j.

A pairwise comparison matrix M is called consistent (or transitive) if

$$m_{ij} * m_{jk} = m_{ik}$$

for every $i, j, k = 1, 2, \ldots, n$.

We will refer to it as a "consistency condition". While every consistent matrix is reciprocal, the converse is false in general. If the consistency condition does not hold, the matrix is inconsistent (or intransitive).

Consistent matrices correspond to the ideal situation in which there are the exact values $s_1, \ldots, s_n$ for the stimuli. The quotients $m_{ij} = s_i/s_j$ then form a consistent matrix. The vector $s = [s_1, \ldots s_n]$ is unique up to a multiplicative constant. The challenge of the pairwise comparisons method comes from the lack of consistency of the pairwise comparisons matrices which arise in practice (while as a rule, all the pairwise comparisons matrices are reciprocal). Given an $N \times N$ matrix M, which is not consistent, the theory attempts to provide a consistent $n \times n$ matrix M which differs from matrix M "as little as possible".

The matrix: $M = s_i/s_j$ is consistent for all (even random) values v_i. It is an important observation since it implies that a problem of approximation is really a problem of a norm selection and the distance minimization. For the Euclidean norm, the vector of geometric means (equal to the principal eigenvector for the transitive matrix) is the one which generates it. Needless to say that only optimization methods can approximate the given matrix for the assumed norm (e.g., LSM for the Euclidean distance, as recently proposed in [8]). Such type of matrix is examined in [18] as "error-free" matrix.

It is unfortunate that the singular form "comparison" is sometimes used considering that a minimum of three comparisons are needed for the method to have a practical meaning. Comparing two entities (stimuli or properties) in pairs is irreducible, since having one entity compared with itself gives trivially 1. Comparing only two entities (2 by 2 PC matrix) does not involve inconsistency. Entities and/or their properties are often called stimuli in the PC research but are rarely used in applications.

3 The pairwise comparisons inconsistency notion

[16] includes: "We may assume that when the inconsistency indicator shows the perturbations from consistency are large and hence the result is unreliable, the information available cannot be used to derive a reliable answer."

The above quotation is consistent with the popular computer adage GIGO (garbage in – garbage out). GIGO summarizes what we have known for a long time: getting good results from "dirty data" is unrealistic, and surely, cannot be guaranteed. An approximation of a pairwise comparisons matrix is meaningful if the inconsistency is acceptable. It can be done by localizing the inconsistency and reducing it to a certain predefined threshold. For the time being, the inconsistency threshold is arbitrary or set by a heuristic, since there is no theory to find it. It is a similar situation to p-value in statistics – often assumed as 0.05 (or any other arbitrary value), but can be undermined for each individual case.

As pointed out earlier, given an inconsistent matrix A, the theory attempts to approximate it with a consistent matrix M that differs from matrix A "as little as possible". The consistency of a matrix A, expressed by $a_{ij} * a_{jk} = a_{ik}$, was called in [16] a "cardinal consistency". In this study, we will use a term "triad" for (a_{ij}, a_{ik}, a_{jk}) (these three matrix elements in the above cardinal consistency condition).

Before we progress to a formal inconsistency definition, the most important question needs to be explained: *"where does the inconsistency come from?"* The short answer to this question is from the excess of input data. The superfluous data comes from collecting data for all pairs combinations which is $n*(n-1)/2$, while only $n-1$ proper comparisons (e.g., the first row or column and even diagonals or some of their combinations) would suffice. The inconsistency in a triad is illustrated by the following example.

Example:

This is an inconsistent matrix M, 3 by 3 with one triad $(2, 2, 2)$, which is marked by the bold font, is:

$$A = \begin{bmatrix} 1 & \mathbf{2} & \mathbf{2} \\ 1/2 & 1 & \mathbf{2} \\ 1/2 & 1/2 & 1 \end{bmatrix}$$

Evidently, matrix A displays an abnormality since $2 * 2 \neq 2$. The common sense dictates that if for "every bar is two times longer than every other bar", all bars should be given equal length. However, the computed vector of weight (s_i mentioned earlier in this section)is:

$$s = [0.4934, 0.3108, 0.1958]$$

Everything comes back to normality when we change $a_{1,3}$ from 2 to 4. Although this is a rather simple example, the proposed inconsistency reduction process comes to finding such a triad and changing an offending value with the value which making the consistency condition to hold or at least to have one side of the consistency condition close to the other side.

Table 1 shows three triads consisting of matrix elements, which may not be neighbors in this matrix. Different types of parenthesis have been used for each triad, only for easier demonstration. All triads above the main diagonal have the carpenter angle tool shape or the mirror image of the capital letter "L", with the middle value in the "elbow" element ideally (for the consistency) being the product of the outer elements.

1		(1,3)				(1,7)
	1		[2,4]		[2,6]	
		1				(3,7)
			1	{4,5}	[4,6]	{4,7}
				1		{5,7}
					1	
						1

Table 1: PC matrix with various triads

Triads may have one overlapping matrix element. For example, $i = 1$, $j = 2$, and $k = 3$ creates a triad with one element in the triad created by $i = 1$, $j = 3$, and $k = 7$. According to the triad production expression: (a_{ij}, a_{ik}, a_{jk}), it is element $a_{1,3}$. Evidently, triad elements do not need to be neighbors in the matrix, but if they are, they must be just above the main diagonal, as illustrated by Table 2.

1	(1,2)	(1,3)				
	1	(2,3)	(2,4)			
		1	(3,4)	(3,5)		
			1	(4,5)	(4,6)	
				1	(5,6)	(5,7)
					1	(6,7)
						1

Table 2: All triads in a 7 by 7 matrix with elements which are neighbors

Inconsistent assessments cannot be accurate but after approximation, they may be closer to real values. Let us assume that the triad $(2, 5, 3)$ in Fig. 1 reflects comparisons of three bars with lengths: A, B, and C made by experts on three different continents by the Internet. Expert 1 compares A to B giving $A/B = 3$ and Expert 2 compares A to B giving $B/C = 2$.

We could object to $A/C = 5$ given by Expert 3 after A to C are compared. Evidently, $A/B * B/C$ is A/C, hence the result is $2 * 3 = 6$. However, we really do not know and will never know who made an estimation error! In fact, we can safely assume that each expert made "just a little bit of error". In particular, none of these three values could be accurate. It cannot be solved by any theory. A solution is needs to be found on individual basis for each application.

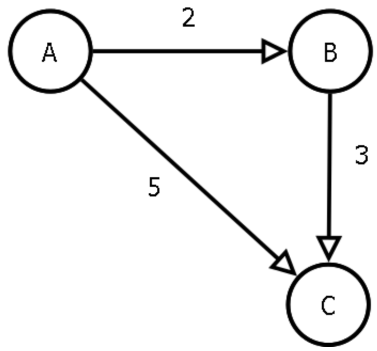

Figure 1: A graphical representation of the triad (2,5,3)

Each triad generates a PC matrix M of the size 3 by 3. Let us use A, B, and C to reflect lengths of three bars. The value $M[1, 2] = 1$ represents $A = B$, $M[2, 3] = 1$ represents $B = C$ hence the expectation is $A = C$ but the third estimates is 5. It is reflected by the last bar hence the error is 500%. As assumed, x can take any arbitrary value and so can the estimation error. Thus, we have made our point and presented the error tolerance for small values of n in Tab. 3. PC matrix with triads $(1, x, 1)$ is of a considerable importance and it is analyzed in Section 6.

4 Axiomatization of inconsistency

It is generally assumed that it was Saaty who in [16] defined PC matrix A as consistent if and only if $a_{ij} * a_{jk} = a_{ik}$ for $i, j, k = 1, 2, ..., n$. However, inconsistency was defined and examined before 1977, by at least these four studies published between 1939 and 1961: [12, 10, 7, 17]. To our knowledge, no axiomatization has ever been proposed for the general case of pairwise comparisons matrix with real positive entries, although it seems that attempts have been made for matrices with integer values for win-tie-loss entries.

The common sense expectations for the inconsistency indicator ii of a triad $T = (x, y, z)$ are:

1. $ii = 0$ for $y = x * z$,

2. $ii \in [0, 1)$ - by common sense, we cannot achieve “ideal inconsistency”,

3. for a consistent triad $ii(x, y, z) = 0$ with $xz = y$, increasing or decreasing x, y, z results in increasing $ii(x, y, z)$.

The third axiom is crucial for any axiomatization. Without this axiom, an inconsistency indicator would not make practical sense. For any assumed definition for inconsistency, an inconsistency indicator of a triad $T' = (x', y', z')$ cannot be smaller than of $T = (x, y, z)$ if it is worse by one of more coordinates, which is what the third axiom is about. That is, $ii(x', y', z') \geq ii(x, y, z)$. It is a reasonable expectation that the worsening of a triad, used in the definition of consistency (also in [16]), cannot make the entire matrix more consistent.
For $ii(x, y, z) > 0$, we have two cases:

(a) $xz < y$

(b) $xz > y$

In case of:

(a) if $x'z' < xz \& y' > y$ then $ii(x, y, z) < ii(x', y', z')$

(b) if $x'z' > xz \& y' < y$ then $ii(x, y, z) < ii(x', y', z')$

Let us look at the following two examples:

- $ii(1.5, 2, 2.5)$ will increase if we increase 1.5 or 2.5, since 1.5*2.5 is already greater than 2. On the other hand, decreasing 2 should also increase the inconsistency.

- $ii(1.5, 2.5, 1.2)$ will increase if we increase 2.5, since it is greater than 1.5*1.2=1.8, but decreasing 1.5 or 1.2 should also increase inconsistency for the same reason.

Based on the proposed axioms for inconsistency and [13], let us define:

$$f(x, y, z) = 1 - \min\{\frac{y}{xz}, \frac{xz}{y}\}$$

.

It is equivalent to:

$$f(x, y, z) = 1 - e^{-|\ln(\frac{y}{xz})|}$$

.

The expression $|\ln(\frac{y}{xz})|$ is the distance of the triad T from 0. When this distance increases, the $f(x, y, z)$ also increases. It is important to notice here that this definition allows us to localize the inconsistency in the matrix PC and it is of a considerable importance for most applications.

Another possible definition of the inconsistency has a global character and needs a bit more explanations. Let $A = \{a_{ij}\}_{i,j=1}^{n}$ be a reciprocal positive matrix. The matrix A is consistent if and only if for any $1 \leq i < j \leq n$ we have:

$$a_{ij} = a_{i,i+1}a_{i+1,i+2} \ldots a_{j-1,j}.$$

Therefore, we may define inconsistency indicator of A as:

$$\mathrm{ii}(A) = 1 - \min_{1\leq i<j\leq n}\left(\min\left(\frac{a_{ij}}{a_{i,i+1}a_{i+1,i+2} \ldots a_{j-1,j}}, \frac{a_{i,i+1}a_{i+1,i+2} \ldots a_{j-1,j}}{a_{ij}}\right)\right)$$

It is equivalent to:

$$\mathrm{ii}(A) = 1 - \max_{i<j}\left(1 - e^{-\left|\ln\left(\frac{a_{ij}}{a_{i,i+1}a_{i+1,i+2}\cdots a_{j-1,j}}\right|\right)}\right)$$

Both *ii* definitions have some advantages and disadvantages. The first definition allows us to find the localization of the inconsistency. The second definition may be useful when the global inconsistency is more important. The first definition follows what is adequately described by the idiom: "one bad apple spoils the barrel". A hybrid of using two definitions may be a practical solution in applications. Alternatively, both definitions can be used in a sequence.

5 The analysis of $CPC(x, n)$ matrix

In this section, we will consider a pairwise matrix with all 1s expect for two corners (hence corner comparisons matrix or CPC). Consider the matrix $CPC(x, n)$, with $x > 1$, defined by

$$CPC(x,n) = \begin{bmatrix} 1 & 1 & \cdots & 1 & x \\ 1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 1 \\ x^{-1} & 1 & \cdots & 1 & 1 \end{bmatrix} \in M_{n\times n}(\mathbb{R})$$

By the Perron-Frobenius theorem, the principal eigenvalue $\lambda_{\max}$ corresponds to a unique (up to constant multiple) eigenvector $w = \{w_i\}_{i=1}^n$ with positive entries. Since the rows $r_2, r_3 \ldots, r_{n-1}$ of the matrix $CPC(x, n)$ are equal the eigenvector, w satisfies $w_2 = w_3 = \ldots = w_{n-1}$. After normalization we may assume that

$$w = (a, 1, 1, \ldots, 1, b).$$

The eigenvalue equation $CPC(x, n)w = \lambda_{\max} w$ is reduced to the system of three equations with three unknown a, b and $\lambda_{\max}$.

$$\begin{aligned} a + n - 2 + bx &= \lambda_{\max} a, \\ a + n - 2 + b &= \lambda_{\max}, \\ \frac{a}{x} + n - 2 + b &= \lambda_{\max} b. \end{aligned}$$

By solving the system consisting of the first and the last linear equations, relative to a and b, we get

$$a = (n-2)\,\frac{x^{-1} + \lambda_{\max} - 1}{\lambda_{\max}^2 - 2\lambda_{\max}}, \quad b = (n-2)\,\frac{x + \lambda_{\max} - 1}{\lambda_{\max}^2 - 2\lambda_{\max}}.$$

Substituting a and b in the second equation by the above expressions (after some transformations), the following third degree equation for $\lambda_{\max}$ is obtained:

$$\lambda_{\max}^3 - n\lambda_{\max}^2 = (n-2)(x^{-1} + x - 2). \tag{1}$$

We can still transform that into

$$\frac{\lambda_{\max} - n}{n-1} = \frac{n-2}{n-1}\,\frac{x^{-1} + x - 2}{\lambda_{\max}^2}.$$

Since the right hand side is positive, we must have $\lambda_{\max} > n$.
Therefore

$$\frac{\lambda_{\max} - n}{n - 1} \leq \frac{n-2}{n-1} \frac{x^{-1} + x - 2}{n^2}. \tag{2}$$

We have assumed that $x > 1$ therefore $x^{-1} < 1$
also

$$\frac{n-2}{n-1} < 1$$

hence the following inequality holds:

$$\frac{\lambda_{\max} - n}{n - 1} \leq \frac{x}{n^2}. \tag{3}$$

The inequality (3) has a very important implication. No matter how large x is, there is always such n that the left hand side of (3) is as small as we wish it to be. So, regardless of the assumed threshold in [16] (de facto, originally set to 10%), the matrix is acceptable according to the consistency rule set in [16].

Evidently, the arbitrarily large x in the matrix $CPC(x, n)$ of size n by n invalidates the acceptability of this matrix. Hence, by a *reductio ad absurdum*, we are must dismiss the soundness of the eigenvalue-based inconsistency indicator represented by the left hand side inequality (3).

Example:

Let $n = 6$ and $x = 6$. We then get

$$\frac{\lambda_{\max} - n}{n - 1} \leq \frac{4}{5} \frac{4 + (1/6)}{36} = 0.0925925...$$

Actually, we can determine numerically that $\lambda_{\max} = 6.406123...$
Then

$$\frac{\lambda_{\max} - n}{n - 1} = 0.081224...$$

Now, we turn to general reciprocal matrices. By a careful analysis of [16], we can get the following lower estimates for $\lambda_{\max}$ for general reciprocal positive matrices.

Theorem 1. *Let $A = \{a_{ij}\}_{i,j}^n$ be a reciprocal matrix with positive entries. Then*

$$\lambda_{\max} \geq n + \frac{3}{n} \frac{\mathrm{ii}^2(A)}{\sqrt[3]{1 - \mathrm{ii}(A)}},$$

where

$$\mathrm{ii}(A) = 1 - \min_{i<k<j} \min \left\{ \frac{a_{ij}}{a_{ik}a_{kj}}, \frac{a_{ik}a_{kj}}{a_{ij}} \right\}.$$

Proof. Let $w = \{w_i\}_{i=1}^n$ be the eigenvector corresponding to the eigenvalue $\lambda_{\max}$. By the Perron-Frobenius theory, we have $w_i > 0$. Thus

$$\lambda_{\max} w_i = \sum_{j=1}^n a_{ij} w_j.$$

By an easy transformation and the fact that $a_{ii} = 1$ (see [16], pages 237-238), we get

$$n\lambda_{\max} - n = \sum_{1 \leq i < j \leq n} \left(a_{ij} \frac{w_j}{w_i} + a_{ji} \frac{w_i}{w_j} \right).$$

This implies

$$n(\lambda_{\max} - n) = \sum_{1 \leq i < j \leq n} \left(a_{ij} \frac{w_j}{w_i} + a_{ji} \frac{w_i}{w_j} - 2 \right) \tag{4}$$

Let us assume that the maximal inconsistency is attained at the triad $s < u < t$, i.e.

$$\mathrm{ii}(A) = 1 - \min \left\{ \frac{a_{st}}{a_{su}a_{ut}}, \frac{a_{su}a_{ut}}{a_{st}} \right\}.$$

Every term in the sum of (4) is nonnegative as $x + x^{-1} - 2 \geq 0$, for $x > 0$ and $a_{ji} = a_{ij}^{-1}$. By reducing the sum to three terms corresponding to the triad $s < u < t$, we get

$$n(\lambda_{\max} - n) \geq a_{su} \frac{w_u}{w_s} + a_{us} \frac{w_s}{w_u} + a_{ut} \frac{w_t}{w_u} + a_{tu} \frac{w_u}{w_t} + a_{st} \frac{w_t}{w_s} + a_{ts} \frac{w_s}{w_t} - 6. \tag{5}$$

Denote

$$x = a_{su} \frac{w_u}{w_s}, \ y = a_{ut} \frac{w_t}{w_u}, \ \alpha = \frac{a_{su}a_{ut}}{a_{st}}.$$

Then the right hand side of (5) is given by

$$f(x, y) := x + x^{-1} + y + y^{-1} + \alpha^{-1}xy + \alpha x^{-1}y^{-1} - 6.$$

By calculating the partial derivatives of $f(x,y)$ and equating them to zero, we can easily determine that the minimal value of $f(x,y)$ is attained for

$$x = y = \alpha^{1/3}.$$

We will consider the case $\alpha \leq 1$, i.e. $\mathrm{ii}(A) = 1 - \alpha$ (the other case $\alpha > 1$ can be dealt with similarly). We have

$$\begin{aligned} f(x,y) &\geq 3(\alpha^{1/3} + \alpha^{-1/3}) - 6 = 3\alpha^{-1/3}(1-\alpha^{1/3})^2 \geq \\ &= 3\alpha^{-1/3}\left(\frac{1-\alpha}{1+\alpha^{1/3}+\alpha^{2/3}}\right)^2 \geq \frac{1}{3}\alpha^{-1/3}(1-\alpha)^2 = \frac{1}{3}\,\frac{\mathrm{ii}^2(A)}{\sqrt[3]{1-\mathrm{ii}(A)}}. \end{aligned}$$

Summarizing we get

$$n(\lambda_{\max} - n) \geq \frac{1}{3}\,\frac{\mathrm{ii}^2(A)}{\sqrt[3]{1-\mathrm{ii}(A)}},$$

which yields the conclusion. □

Remark. Theorem 1 yields

$$\frac{\lambda_{\max} - n}{n-1} \geq \frac{1}{3(n-1)n}\,\frac{\mathrm{ii}^2(A)}{\sqrt[3]{1-\mathrm{ii}(A)}}.$$

Thus for given n (say $n = 6$), the quantity explodes if the indicator $\mathrm{ii}(A)$ approaches the value 1.

We can obtain another lower estimate for $\lambda_{\max}$, as well which takes into account the total inconsistency information of the matrix A.

Theorem 2. *Let T denote the set of all triads in the matrix A and $\mathrm{ii}(t)$ be the inconsistency indicator of the triad t, i.e. for $t = (i,k,j)$ with $i < k < j$, let*

$$\mathrm{ii}(t) = 1 - \min\left\{\frac{a_{ij}}{a_{ik}a_{kj}}, \frac{a_{ik}a_{kj}}{a_{ij}}\right\}.$$

Then

$$\lambda_{\max} \geq n + \frac{1}{3n(n-2)}\sum_{t\in T}\frac{\mathrm{ii}^2(t)}{\sqrt[3]{1-\mathrm{ii}(t)}}.$$

Proof. Every term a_{uv} with $1 \leq u < v \leq n$ belongs to $n-2$ triads. Therefore the formula (4) implies

$$(n-2)n\,(\lambda_{\max} - n) = \sum_{i<k<j} \left[a_{ik}\frac{w_k}{w_i} + a_{ki}\frac{w_i}{w_k} + a_{kj}\frac{w_j}{w_k} + a_{jk}\frac{w_k}{w_j} + a_{ij}\frac{w_j}{w_i} + a_{ji}\frac{w_i}{w_j} - 6 \right].$$

By the proof of Theorem 1 we get that for $\alpha = \min\{a_{ik}a_{kj}/a_{ij},\, a_{ij}/a_{ik}a_{kj}\}$ and $t = (i,k,j)$ we have

$$a_{ik}\frac{w_k}{w_i} + a_{ki}\frac{w_i}{w_k} + a_{kj}\frac{w_j}{w_k} + a_{jk}\frac{w_k}{w_j} + a_{ij}\frac{w_j}{w_i} + a_{ji}\frac{w_i}{w_j} - 6 \geq \frac{1}{3}\alpha^{-1/3}(1-\alpha)^2 = \frac{1}{3}\,\frac{\mathrm{ii}^2(t)}{\sqrt[3]{1-\mathrm{ii}(t)}}.$$

Hence

$$(n-2)n\,(\lambda_{\max} - n) \geq \frac{1}{3}\sum_{t\in T} \frac{\mathrm{ii}^2(t)}{\sqrt[3]{1-\mathrm{ii}(t)}}.$$

□

The $CPC(x,n)$ matrix in the above example shows that the eigenvalue-based consistency index (CI) tolerates error of an arbitrary value for the large enough n (the matrix size). According to AHP theory, the $CPC(x,n)$ matrix is considered "consistent enough" (or "good enough") for $CI \leq 0.1$, although it has n arbitrarily erroneous elements in it. The number n of the erroneous elements grow to infinity with the growing n and it invalidates using CI for measuring the inconsistency.

5.1 The interpretation of the $CPC(x,n)$ analysis

Matrix $CPC(x,n)$ of the size of 3 by 3 has only one triad: $(1,x,1)$. Trivially, the only value of x for this matrix to be consistent is 1 ($x = 1 * 1$). For $x = 2.62$, we have:

$$A3 = \begin{bmatrix} 1 & 1 & 2.62 \\ 1 & 1 & 1 \\ 0.381679389 & 1 & 1 \end{bmatrix}$$

The principal eigenvalue of $A3$ is 3.10397 hence $CI = 0.051985$ and it is less than 10% of $RI = 0.52$, hence acceptable due to the fact that the proposed consistency index (CI) is defined in [16] as:

$$CI = \frac{\lambda_{max} - n}{n - 1}$$

and the consistency ratio (CR) defined as

$$CR = \frac{CI}{RI}$$

where RI is the average value of CI for random matrices and computed as 0.52 (decreased from 0.58 as stipulated in [16]).

As we previously observed, x should be 1, so $x = 2.62$ gives us 262% error and it is what we call the tolerance error since for matrices 3 by 3, RI has been computed as 0.5245 hence $CR < 0.1$ for $CPC(2.62, 3)$. The tolerance error, for other n from 3 to 7 has been computed and presented in Tab. 3

Table 3: Error tolerance of eigenvalue-based inconsistency for $CPC(x, n)$

n	error tolerance for (1,x,1)
3	262%
4	417%
5	618%
6	875%
7	1,170%

$CPC(x, n)$ of the size n by n has $n - 2$ triads of this shape: $(1, x, 1)$. All triads are formed from these matrix elements (a_{ij}, a_{ik}, a_{jk}) based the consistency condition is $a_{ik} = a_{ij} * a_{jk}$. Not only the equality does not hold for $x > 1$ but for $a_{ij} = a_{jk} = 1$ and $x = a_{ij} * a_{jk}$ the inaccuracy grows with the growing x. For $CPC(2.62, 3)$, it is illustrated by Fig. 2. The question is evident: "Would you consider such three bars are equal?" and if the answer is not, "why is it acceptable for AHP to tolerate such error?"

Values x can be an arbitrarily large value which creates a problem. Assuming that the exact values are set to $a_{ij} = a_{jk} = 1$, the value x is computed as $a_{ij} * a_{jk} = 1$ hence the estimation error for x is $x/(1*1)$ hence x or $x*100\%$. For example, for $n = 7$, $x = 4.25$ giving the tolerance error 1,170%. However,

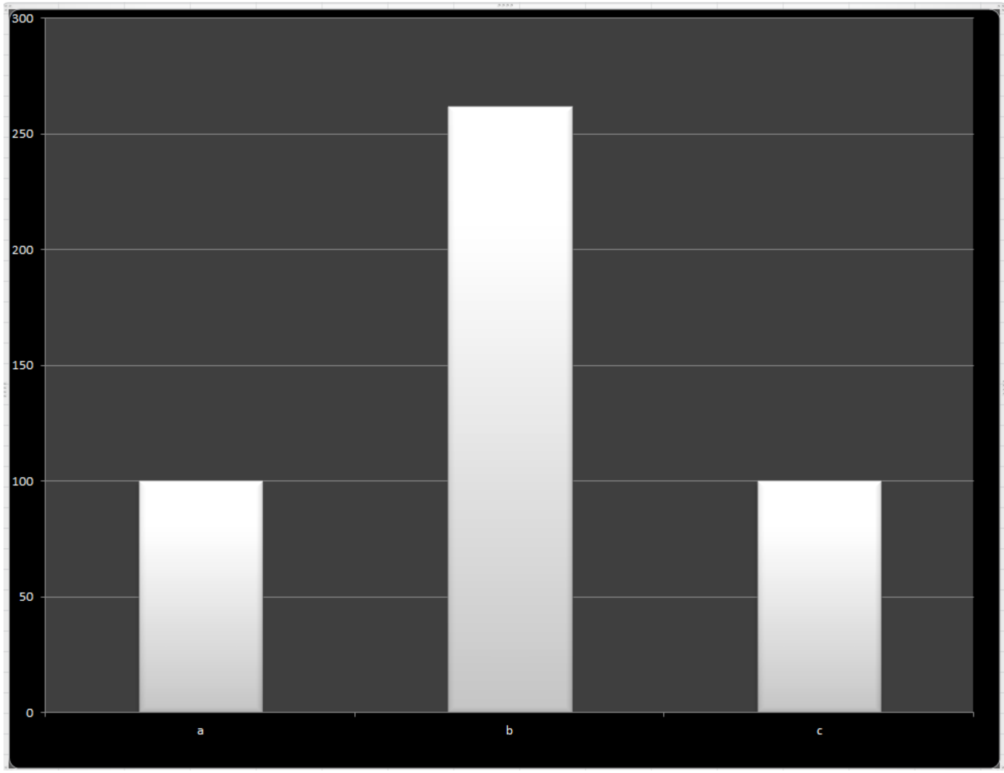

Figure 2: Triad (a, b, c) with the 262% error tolerated by the eigenvalue-based inconsistency for $CPC(2.62, 3)$

x can be 1,000,000%, or more since in Section 6, we have provided a proof that there is such n for which $CI \leq 0.1$ hence acceptable. The 10% threshold, originally set as "the consistency rule" in [16] and later on slightly decreased for larger n but it does not matter for the inequality (3) in Section 6 if it is 10% or any other fixed value.

According the the results in Section 6, there is always such n for which the deviation of the principal eigenvalue from n is small enough to consider $CPC(x, n)$ matrix acceptable while the arbitrarily large x has $n - 2$ triads with an unacceptably high error x.

The distance-based inconsistency was introduced in [13] and independently analyzed in [2]. Its convergence analysis was published in [15]. Evidently, it does not tolerates big values of x in triads $(1, x, 1)$. It specifically postulates to re-examine input data for $ii > 1/3$, hence $x > 1.5$ is proclaimed

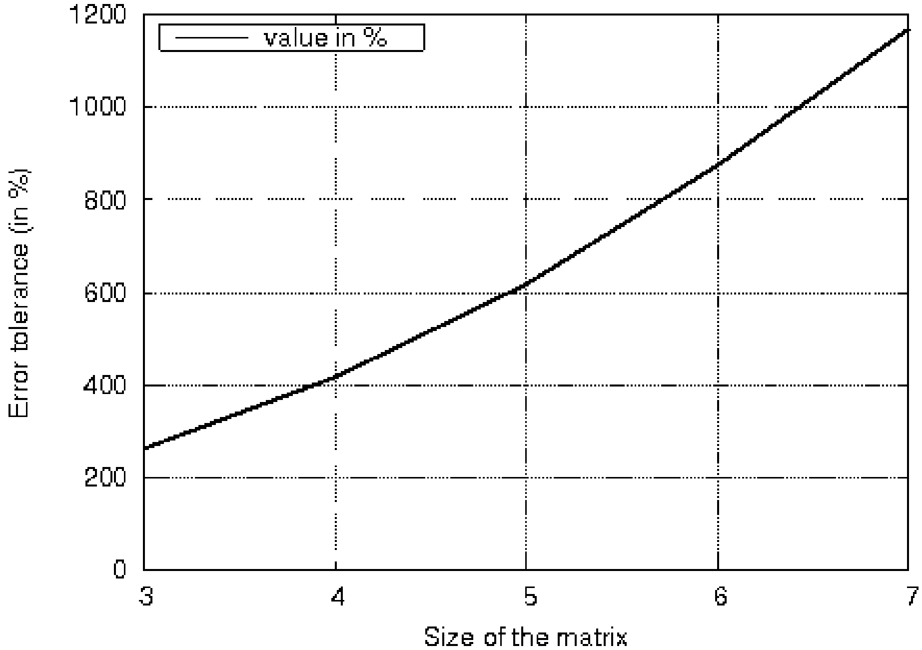

Figure 3: Error tolerance of eigenvalue-based inconsistency for $CPC(x,n)$

to be suspiciously high and the PC matrix needs to be re-examined.

6 The analysis of $FPC(x,n)$ matrix

We fear that some of the AHP supporters may hold to the last hope by believing that "it is only one value in the $CPC(x,n)$ matrix" since it has x in one matrix element (in fact, x^{-1} in another corner). However, we have a surprise for them by what we call FPC (the "full" pairwise comparisons matrix or the PC matrix full of x). Unlike $CPC(x,n)$, it has all erroneous triads.

Consider the matrix $FPC(x,n)$, with $x > 1$, defined by

$$FPC(x,n) = \begin{bmatrix} 1 & x & \cdots & x & x \\ x^{-1} & 1 & \cdots & x & x \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x^{-1} & x^{-1} & \cdots & 1 & x \\ x^{-1} & x^{-1} & \cdots & x^{-1} & 1 \end{bmatrix} \in M_{n\times n}(\mathbb{R})$$

Let w be the eigenvector corresponding to the principal eigenvalue λ_{max}. Thus

$$x^{-1}(w_1 + \ldots + w_{k-1}) + w_k + x(w_{k+1} + \ldots + w_n) = \lambda w_k$$

for $k = 1, 2, \ldots, n$.

We notice that for $k = 1$, the first term is missing while for $k = n$, the last term is missing. By subtracting equations corresponding to k and $k-1$, we get the following:

$$x^{-1}w_{k-1} + w_k - w_{k-1} - xw_k = \lambda w_k - \lambda w_{k-1}$$

which gives

$$w_k = w_{k-1}\frac{x^{-1} - 1 + \lambda}{x - 1 + \lambda}$$

for $k = 2, \ldots, n$.

hence

$$w_k = \left(\frac{x^{-1} - 1 + \lambda}{x - 1 + \lambda}\right)^{k-1}$$

for $k = 1, 2, \ldots, n$.

Substituting it into the first equation results in

$$1 + x(w_2 + w_2^2 + \ldots + w_2^{n-1}) = \lambda$$

hence

$$1 + x\frac{w_2^n - w_2}{w_2 - 1} = \lambda$$

by using

$$w_2 = \frac{x^{-1} - 1 + \lambda}{x - 1 + \lambda}$$

and by transforming the last equation, we get

$$\left(\frac{x^{-1} - 1 + \lambda}{x - 1 + \lambda}\right)^n = \frac{1}{x^2}$$

therefore

$$\lambda = \frac{x-1}{x}\frac{x+x^{\frac{2}{n}}}{x^{\frac{2}{n}}-1}$$

Example:

For $x = 2.25$ and $n = 4$, we have $\lambda_{max} = \frac{25}{6}$
Thus

$$\frac{\lambda_{max} - n}{n-1} = \frac{\frac{25}{6} - 4}{3} = \frac{1}{18} \approx 0.055555556$$

therefore 225% error is tolerated by AHP theory for $n = 4$. We leave to the reader the soundness of entering three inaccurate (by 55.6%) comparisons into the matrix $FPC(x, n)$ and claiming that such matrix is acceptable.
For $x = 2.84$ and $n = 7$, the tolerated error increases to 64.79%. These tolerated errors although a bit less impressive than for $CPC(x, n)$ are still by far too high for the estimation lengths of randomly generated bars as it was demonstrated by a Monte Carlo Study in [14] where a 5% error was reported. The error 284% is bigger than 262% illustrated in Fig.2. The same question remains about error tolerance acceptableness. The authors of this study consider it unacceptable. The question is if you consider three bars in Fig.2 as equal enough. The only equality of this kind, which comes to our minds is: "All animals are equal, but some animals are more equal than others." [George Orwell, Animal Farm].

7 Conclusions

The presented inconsistency axiomatization is simple, elegant, a considerable step forward and a sound mathematical foundation for the further PC research. It finally allows us to define proper inconsistency indicators, regardless of whether or not they are localizing the inconsistency or serve as global indicators of inconsistencies in pairwise comparisons matrices. The distance-based inconsistency definition localizes inconsistency and produces correct results.

The eigenvalue-based consistency index (CI) fails to increase with the growing size of the PC matrix and tolerates the growing number of triads with each of them having an unacceptable level of inconsistency. As proven in Section 6, AHP thresholds (both old and recently modified) are unable to detect large quantities of large inaccuracies existing in $CPC(x, n)$ matrices. There is always n, for which these inaccuracies are lost in the matrix, no matter how large they are. The discussed eigenvalue-based inconsistency indicator is not precise enough for the detection of individual triads, which turns to be erroneous but "averaged" by the eigenvalue processing. It is anticipated that every statistical inconsistency indicator, including those with roots in the principal eigenvalue, may not be good indicators of the problems existing in pairwise comparisons. Simply, they to not look deep enough into relationships existing in cycles of which triads are the most important minimal cycles (as pointed out in this study, one or two elements cannot create an inconsistency cycle). Hopefully, proponents of other inconsistency indicators will examine their definition by using the proposed axiomatization. Certainly, getting help from authors of this study is a vital solution.

During the final stages of editing of our study for publication, the numerical results strongly supporting our finding were located in [20] with the following text in the conclusions:

> "In this paper, by simulation analysis, we obtain the following result: as the matrix size increases, the percent of the matrices with acceptable consistency ($CR \leq 0.1$), decrease dramatically, but, on the other hand, there will be more and more contradictory judgments in these sufficiently consistent matrices. This paradox shows that it is impossible to find some proper critical values of CR for different matrix sizes. Thus we argue that Saaty's consistency test could be unreasonable."

It is not a paradox anymore. In this study, we have provided a mathematical proof and reasoning for it.

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