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Inconsistency indicator maps on groups for pairwise comparisons ☆

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ABSTRACT

This study presents an abelian group approach to analyzing inconsistency in pairwise comparisons. A notion of an inconsistency indicator map on a group, taking values in an abelian linearly ordered group, is introduced. For it, metrics and generalized metrics are utilized. Every inconsistency indicator map generates both a metric on a group and an inconsistency indicator of an arbitrary pairwise comparisons matrix over the group.

1. Introduction

The first documented use of pairwise comparisons (PC) is attributed to Ramon Llull, a 13th-century mystic and philosopher. Thurstone applied pairwise comparisons to experimental psychology and delivered the first formal introduction of pairwise comparisons in the form of "the law of comparative judgement" in [17]. A variation of this law is known as the BTL (Bradley–Terry–Luce) model (cf. [3]). A number of controversial customized pairwise comparisons have been considered in numerous studies. However, we do not intend to support any customization here. Amongst many others, Saaty's seminal work [16] had a considerable impact on the pairwise comparisons (PC) research. The authors' position is that the influence of [16] on the pairwise comparisons research should be acknowledged despite serious controversies generated by it. This work provides a more refined alo-group perspective on inconsistency in pairwise comparisons originated in [2] where deficiencies exist and some (but not all) are addressed in this study.

All measurements (physical or not) are related to pairwise comparisons. For example, when we say that the distance between stars A and B is 2.71 light years, we simply compare the unit of distance (in this case, one light year) to the distance between A and B. Pairwise comparisons are of particular use where a unit of measurement cannot be defined and it is so for most subjective assessments. For example, public safety or environmental pollution lacks a unit (or a "yard

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stick") for the measurement, however, it is still necessary to measure it by comparing them to each other and expressing it by a ratio stored in what we call a PC matrix.

A triad (x, y, z) of entries of PC matrix, expressing ratios of three entities, is called consistent if $xz = y$. This is a case of *multiplicative consistency* expressing ratios of compared entities. Another approach to the consistency of triads of numbers is additive. Namely, a triad x, y, z of numbers is called *additively consistent* if $x + z = y$. Additive consistency is relevant to the question: "by how much x is larger (or more important) than y". In most studies about PC, the multiplicative consistency has been mostly considered (e.g., [1]). Additive PC have been analyzed in [19] recently. Differences between quality values instead of ratios appear in BTL model. In [2], both multiplicative and additive pairwise comparisons were unified to the comparisons of elements of abelian linearly ordered groups (in abbreviation; alo-groups). A consistency index of triads of elements of an alo-group was defined in [2] (for a single triad, defined in [8] of 1993). An inconsistency indicator for triads, introduced in [8] for the multiplicative case in 1993 for a single triad was extended in [4]. The analysis of Koczkodaj's inconsistency indicator is included in [1] and [11]. An axiomatization of inconsistency in pairwise comparisons was proposed in [11] (with some minor deficiencies being currently corrected).

The main aim of this work is to introduce and investigate a new general concept of an inconsistency indicator map on a group, strictly relevant to a generalized metric which takes values in an alo-group. The inconsistency indicator maps induce inconsistency indicators of pairwise comparisons matrices.

Axioms of logic, rules of deduction and system ZF are assumed in this work. The notation and terminology of [12] is followed. In particular, ω is the set of all non-negative integers (of von Neumann), $0 = \emptyset$, $1 = \{0\}$ and $n + 1 = n \cup \{n\}$ for each $n \in \omega$ (cf. [14]).

2. A PC matrix over a group

Let $X = \langle X, \cdot \rangle$ be a group. We denote by 1_X or, for simplicity, by 1 the unit element of X. For $a \in X$, the symmetric (inverse) element of a in X is denoted by a^{-1}.

Let K be a non-void finite set. A $K \times K$ matrix $A = [a_{i,j}]$ over X is a mapping $A : K \times K \to X$ such that $a_{i,j} = A(i, j)$ for each pair $\langle i, j \rangle \in K \times K$. There exists $n \in \omega$ such that the sets n and K are equipollent. Let $\psi : n \to K$ be a bijection. We can define an $n \times n$ matrix A_ψ over X by $A_\psi(i, j) = A(\psi(i), \psi(j))$ for each $\langle i, j \rangle \in n \times n$. Therefore, without loss of generality, we can concentrate on $n \times n$-matrices where $n \in \omega \setminus \{0\} = \{1, 2, \ldots\}$.

Definition 2.1. For $n \in \omega \setminus \{0\}$, let $A = [a_{i,j}]$ be an $n \times n$ matrix such that $a_{i,j} \in X$ for all $i, j \in n$. We say that:

(i) the matrix A is a *pairwise comparisons matrix* (in abbreviation a *PC matrix*) *over the group* X if $a_{i,i} = 1_X$ and $a_{i,j} = a_{j,i}^{-1}$ for all $i, j \in n$;
(ii) the matrix A is a *consistent matrix over the group* X if $a_{i,k} \cdot a_{k,j} = a_{i,j}$ for all $i, j, k \in n$.

Remark 2.2. If $\cdot$ is the standard multiplication of positive real numbers, we call the ordered pair $\mathbb{R}_+ = \langle (0; +\infty), \cdot \rangle$ the *standard multiplicative group of positive real numbers*. The notion of a PC matrix over this group coincides with the usual notion of a PC matrix used by many PC researchers. Pairwise comparisons matrices over a group equipped with a linear order were also considered in [2].

Fact 2.3. *Every consistent $n \times n$ matrix $A = [a_{i,j}]$ over a group X is a PC matrix over X.*

3. An inconsistency indicator map on a group

To formulate a definition of an inconsistency indicator map, we will use the following notion of an alo-group investigated in [2] and, for example, also in [7]:

Definition 3.1. An *abelian linearly ordered group* (abbreviated to "alo-group") is an ordered pair $\langle \langle G, \odot \rangle, \leq \rangle$ where $\langle G, \odot \rangle$ is an abelian group, while $\leq$ is a linear order on G such that if $a, b, c \in G$ and $a \leq b$, then $a \odot c \leq b \odot c$.

Distance functions taking values in alo-groups were considered, for instance, in [2] and [7]. We modify Definition 3.2 of [2], by dropping its first condition $d(a, b) \geq e$ and changing G to X in the domain of d, to the following:

Definition 3.2. Let $\mathcal{G} = \langle \langle G, \odot \rangle, \leq \rangle$ be an alo-group. Let 1_G be the neutral element of $\langle G, \odot \rangle$. A *$\mathcal{G}$-metric* or a *$\mathcal{G}$-distance* on a set X is a function $d : X^2 \to G$ such that, for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x, y) = 1_G \Leftrightarrow x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq d(x, z) \odot d(z, y)$.

The first condition $d(a,b) \geq e$ (where $e = 1_G$ in our notation) of Definition 3.2 of [2] is excessive and it becomes a rather simple Proposition 3.3 (it is included here for completeness although it might have been buried in some publication).

Proposition 3.3. *Let $\mathcal{G} = \langle\langle G, \odot\rangle, \leq\rangle$ be an alo-group and let d be a $\mathcal{G}$-metric on a set X. Therefore, for all $x, y \in X$, the following inequality holds: $1_G \leq d(x, y)$.*

Proof. Let $x, y \in X$ and let $a = d(x, y)$. We have $1_G = d(x, x) \leq d(x, y) \odot d(y, x) = a \odot a$, so $a^{-1} \leq a$. Suppose that $a < 1_G$. Therefore, $1_G = a \odot a^{-1} < a^{-1} \leq a$, which is a contradiction. Hence $1_G \leq a$. $\square$

In what follows, let us assume that $\mathcal{G} = \langle\langle G, \odot\rangle, \leq\rangle$ is an alo-group.

Fact 3.4. *Let $\Phi : X \to Y$ be an injection from a set X to a set Y. If $d : Y^2 \to G$ is a $\mathcal{G}$-metric on the set Y, then $\rho : X^2 \to G$, given by the formula $\rho(x, y) = d(\Phi(x), \Phi(y))$ for all $x, y \in X$, is a $\mathcal{G}$-metric on the set X.*

Definition 3.5. (Cf. Definition 3.1 of [2].)

(i) The function $\| \cdot \| : G \to G$, defined by $\| x \| = \max\{x, x^{-1}\}$ for each $x \in X$, is called the *$\mathcal{G}$-norm* on G.
(ii) Let $d_{\mathcal{G}}(x, y) = \| x \odot y^{-1} \|$ for all $x, y \in G$. Therefore, the function $d_{\mathcal{G}} : G^2 \to G$ is called the *$\mathcal{G}$-metric induced by the $\mathcal{G}$-norm on G*.

Notice that, by Proposition 3.2 of [2], $d_{\mathcal{G}}$ is a $\mathcal{G}$-metric on the set G, indeed. Let us introduce a new concept of a $\mathcal{G}$-inconsistency indicator map.

Definition 3.6. Let $X = \langle X, \cdot\rangle$ be a group. A *$\mathcal{G}$-distance-based inconsistency indicator map* (in abbreviation: a $\mathcal{G}$-inconsistency indicator map) on the group X is a function $T : X^3 \to G$ such that, for all $a, b, c, d, e \in X$, the following conditions are satisfied:

(i) $T(a, b, c) = 1_G \Leftrightarrow ac = b$;
(ii) $T(a, b, c) = T(b, ac, 1)$;
(iii) $T(a, de, c) \leq T(a, b, c) \odot T(d, b, e)$.

Fact 3.7. *Let $\Phi : \langle X, +\rangle \to \langle Y, \cdot\rangle$ be an isomorphism of a group $\langle X, +\rangle$ onto a group $\langle Y, \cdot\rangle$. Suppose that $T : Y^3 \to G$ is a $\mathcal{G}$-inconsistency indicator map on $\langle Y, \cdot\rangle$. Therefore, $S : X^3 \to G$, given by the formula $S(a, b, c) = T(\Phi(a), \Phi(b), \Phi(c))$ for all $a, b, c \in X$, is a $\mathcal{G}$-inconsistency indicator map on $\langle X, +\rangle$.*

Proposition 3.8. *Let T be a $\mathcal{G}$-inconsistency indicator map on a group X and let $d_T(x, y) = T(x, y, 1)$ for all $x, y \in X$. Therefore, $d_T : X^2 \to G$ is a $\mathcal{G}$-metric on X such that, for all $x, y, z \in X$, the equality $d_T(xz, y) = T(x, y, z)$ holds.*

Proof. Let $x, y, z \in X$. We will refer to conditions (i)–(iii) of Definition 3.6. By (i), we have: $d_T(x, y) = 1_G \Leftrightarrow T(x, y, 1) = 1_G \Leftrightarrow x = y$. In view of (ii), $d_T(x, y) = T(x, y, 1) = T(x, y \cdot 1, 1) = T(y, x, 1) = d_T(y, x)$. Moreover, it follows from (iii) that $d_T(x, y) = T(x, y, 1) = T(x, y \cdot 1, 1) \leq T(x, z, 1) \odot T(y, z, 1) = d_T(x, z) \odot d_T(z, y)$. In consequence, d_T is a $\mathcal{G}$-metric on X. In the light of (ii), $d_T(xz, y) = T(xz, y, 1) = T(xz, y \cdot 1, 1) = T(y, xz, 1) = T(x, y, z)$. $\square$

Definition 3.9. Let T be a $\mathcal{G}$-inconsistency indicator map on a group X. Therefore, the function $d_T : X^2 \to \mathbb{R}$, defined by $d_T(x, y) = T(x, y, 1)$ for all $x, y \in X$, is called the *$\mathcal{G}$-metric induced by T*.

Proposition 3.10. *Every $\mathcal{G}$-inconsistency indicator map T on a group X satisfies the following conditions:*

(i) *$1_G \leq T(x, y, z)$ for all $x, y, z \in X$;*
(ii) *the group X is abelian if and only if $T(x, y, z) = T(z, y, x)$ for all $x, y, z \in X$.*

Proof. Assume that T is a $\mathcal{G}$-inconsistency indicator map on a group X. Let $x, y, z \in X$. By Proposition 3.8, we have $T(x, y, z) = d_T(xz, y)$ where d_T is the $\mathcal{G}$-metric induced by T. Hence (i) holds in view of Proposition 3.3. Suppose that X is not abelian. There exist $a, b \in X$ such that $ab \neq ba$. Therefore, $T(a, ab, b) = d_T(ab, ab) = 1_G$, while $T(b, ab, a) = d_T(ba, ab) \neq 1_G$. This completes the proof. $\square$

Proposition 3.11. *Let X be a group and let $d : X^2 \to G$ be a $\mathcal{G}$-metric. Therefore, the function $T : X^3 \to G$, defined by $T(x, y, z) = d(xz, y)$ for all $x, y, z \in X$, is a $\mathcal{G}$-inconsistency indicator map on X such that $d = d_T$.*

Proof. Let $a, b, c, d, e \in X$. We have: $T(a, b, c) = 1_G \Leftrightarrow d(ac, b) = 1_G \Leftrightarrow ac = b$. Moreover, $T(a, b, c) = d(ac, b) = d(b, ac) = d(b \cdot 1, ac) = T(b, ac, 1)$. Finally, $T(a, de, c) = d(ac, de) \leq d(ac, b) \odot d(b, de) = T(a, b, c) \odot d(de, b) = T(a, b, c) \odot T(d, b, e)$. This proves that T is a $\mathcal{G}$-inconsistency indicator map on X. It is obvious that $d = d_T$. □

Definition 3.12. For a $\mathcal{G}$-metric d on a group X, the *$\mathcal{G}$-inconsistency indicator map T_d on X induced by d* is defined by $T_d(x, y, z) = d(xz, y)$ for all $x, y, z \in X$.

Theorem 3.13. *Let X be a group. A function $T : X^3 \to G$ is a $\mathcal{G}$-inconsistency indicator map on X if and only if the function $d_T : X^2 \to G$, defined by $d_T(x, y) = T(x, y, 1)$ for all $x, y \in X$, is a $\mathcal{G}$-metric on X such that $d_T(xz, y) = T(x, y, z)$ for all $x, y, z \in X$.*

Proof. This theorem follows easily from Propositions 3.8 and 3.11 when combined together. □

Corollary 3.14. *If $a \in G$ and $1_G < a$, while T_1, T_2 are $\mathcal{G}$-inconsistency indicator maps on a group X, then the functions $\max\{T_1, T_2\}$, $T_1 \odot T_2$ and $\min\{T_1, a\}$ are all inconsistency indicator maps on X.*

Proof. Let d_i be the $\mathcal{G}$-metric induced by T_i for $i \in \{1, 2\}$. It is not difficult to check that the functions $\max\{d_1, d_2\}, d_1 \odot d_2$ and $\min\{d_1, a\}$ are $\mathcal{G}$-metrics. For instance, $\min\{d_1, a\}$ is a $\mathcal{G}$-metric can be shown similarly to Theorem 4.1.3 of [5] if we replace 1 by a and $+$ by $\odot$ in the proof to Theorem 4.1.3 in [5]. Since

$$\max\{T_1(x, y, z), T_2(x, y, z)\} = \max\{d_1(xz, y), d_2(xz, y)\},$$

$$T_1(x, y, z) \odot T_2(x, y, z) = d_1(xz, y) \odot d_2(xz, z)$$

and $\min\{T_1(x, y, z), a\} = \min\{d_1(xz, y), a\}$, it suffices to apply Theorem 3.13 to conclude the proof. □

Definition 3.15. We say that a *$\mathcal{G}$-inconsistency indicator map T on a group X is bounded by $a \in G$* if $T(x, y, z) \leq a$ for all $x, y, z \in X$.

Proposition 3.16. *Let $\langle X_1, +\rangle$, $\langle X_2, \cdot\rangle$ be groups and let $X = X_1 \times X_2$ be equipped with the product operation $\langle x_1, x_2\rangle * \langle y_1, y_2\rangle = \langle x_1 + y_1, x_2 \cdot y_2\rangle$ for $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$. Suppose that T_i is a $\mathcal{G}$-inconsistency indicator map on the group X_i for $i \in \{1, 2\}$. For all elements $\langle x_1, x_2\rangle, \langle y_1, y_2\rangle, \langle z_1, z_2\rangle$ of X, we define*

$$T(\langle x_1, x_2\rangle, \langle y_1, y_2\rangle, \langle z_1, z_2\rangle) = \max\{T_1(x_1, y_1, z_1), T_2(x_2, y_2, z_2)\}.$$

Therefore, the mapping $T : X^3 \to G$ is a $\mathcal{G}$ inconsistency indicator map on the group X.

Proof. Let d_i be the $\mathcal{G}$-metric induced by T_i for $i \in \{1, 2\}$. Put

$$d(\langle x_1, x_2\rangle, \langle y_1, y_2\rangle) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

for all elements $\langle x_1, x_2\rangle, \langle y_1, y_2\rangle$ of X. Therefore d is a $\mathcal{G}$-metric on X such that

$$T(\langle x_1, x_2\rangle, \langle y_1, y_2\rangle, \langle z_1, z_2\rangle) = d(\langle x_1, x_2\rangle * \langle z_1, z_2\rangle, \langle y_1, y_2\rangle)$$

for all elements $\langle x_1, x_2\rangle, \langle y_1, y_2\rangle, \langle z_1, z_2\rangle$ of X. To conclude the proof, it suffices to use Proposition 3.11. □

Proposition 3.17. *Let T be a $\mathcal{G}$-inconsistency indicator map on a group X. Therefore, the function $S : X^3 \to G$, defined by $S(x, y, z) = T(z, y, x)$ for all $x, y, z \in X$, is a $\mathcal{G}$-inconsistency indicator map on X if and only if X is abelian.*

Proof. If X is abelian, then $S = T$ by Proposition 3.10(ii). Now, assume that S is a $\mathcal{G}$-inconsistency indicator map on X and define $F = \max\{T, S\}$. By Corollary 3.14, F is a $\mathcal{G}$-inconsistency indicator map on X. Since $F(x, y, z) = \max\{T(x, y, z), S(x, y, z)\} = \max\{T(x, y, z), T(z, y, x)\} = F(z, y, x)$ for all $x, y, z \in X$, we deduce from Proposition 3.10(ii) that X is abelian. □

Proposition 3.18. *Let T be a $\mathcal{G}$-inconsistency indicator map on a group X. For $x, y, z \in X$, let $T_{-1}(x, y, z) = T(z^{-1}, y^{-1}, x^{-1})$. Therefore $T_{-1} : X^3 \to G$ is a $\mathcal{G}$-inconsistency indicator map on X.*

Proof. Let d_T be the $\mathcal{G}$-metric induced by T. For $x, y \in X$, we define $d(x, y) = d_T(x^{-1}, y^{-1})$. Therefore, $d : X^2 \to G$ is a $\mathcal{G}$-metric on X. By Proposition 3.11, the mapping $T_d : X^3 \to G$, defined by $T_d(x, y, z) = d(xz, y)$ for all $x, y, z \in X$, is a $\mathcal{G}$-inconsistency indicator map on X. We observe that $d(xz, y) = d_T(z^{-1}x^{-1}, y^{-1}) = T(z^{-1}, y^{-1}, x^{-1}) = T_{-1}(x, y, z)$. Hence T_{-1} is the $\mathcal{G}$-inconsistency indicator map induced by d. □

Definition 3.19. Let T be a $\mathcal{G}$-inconsistency indicator map on a group X. The *pairwise symmetrization of* T is the mapping $T^s : X^3 \to G$ defined by

$$T^s(x, y, z) = \max\{T(x, y, z), T(z^{-1}, y^{-1}, x^{-1})\}$$

for all $x, y, z \in X$.

Definition 3.20. A $\mathcal{G}$-inconsistency indicator map T on a group X is called *pairwise symmetric* if $T(x, y, z) = T(z^{-1}, y^{-1}, x^{-1})$ for all $x, y, z \in X$.

Fact 3.21. *A $\mathcal{G}$-inconsistency indicator map T on a group X is pairwise symmetric if and only if $T = T^s$.*

Proposition 3.22. *If T is a $\mathcal{G}$-inconsistency indicator map on a group X, then the pairwise symmetrization T^s of T is a pairwise symmetric $\mathcal{G}$-inconsistency indicator map on X.*

Proof. It suffices to apply Corollary 3.14 and Proposition 3.18. □

Definition 3.23. If $\mathcal{G}$ is $\langle\langle\mathbb{R}, +\rangle, \le\rangle$, where $+$ is the standard addition of real numbers and $\le$ is the standard linear order in $\mathbb{R}$, then every $\mathcal{G}$-inconsistency indicator map on a group X will be called a *real inconsistency indicator map* or *an inconsistency indicator map* on X. We call $\langle\langle\mathbb{R}, +\rangle, \le\rangle$ the *additive real alo-group*.

The following definition generalizes the notion of the $\mathcal{G}$-norm given in [2] and recalled in our Definition 3.5.

Definition 3.24. A *$\mathcal{G}$-absolute value* on a group X is a function $v : X \to G$ such that, for all $x, y \in X$, the following conditions are satisfied:

(i) $1_G \le v(x)$;
(ii) $v(x) = 1_G \Leftrightarrow x = 1$;
(iii) $v(x \cdot y) \le v(x) \odot v(y)$.

If $\mathcal{G}$ is the additive real alo-group, then a $\mathcal{G}$-absolute value on X will be called a *real absolute value* on X.

In view of Proposition 3.1 of [2], the $\mathcal{G}$-norm is a $\mathcal{G}$-absolute value on G.

Fact 3.25. *Let $\Phi : \langle X, +\rangle \to \langle Y, \cdot\rangle$ be an isomorphism of groups and let $v : Y \to G$ be a $\mathcal{G}$-absolute value on $\langle Y, \cdot\rangle$. Then $w = v \circ \Phi : X \to G$ is a $\mathcal{G}$-absolute value on $\langle X, +\rangle$.*

Fact 3.26. *If v is a $\mathcal{G}$-absolute value on a group X, then the function $d_v : X^2 \to \mathcal{G}$, defined by $d_v(x, y) = \max\{v(x \cdot y^{-1}), v(y \cdot x^{-1})\}$ for all $x, y \in X$, is a $\mathcal{G}$-metric on X.*

Definition 3.27. When v is a $\mathcal{G}$-absolute value on X, then the function d_v, defined in Fact 3.26, will be called *the $\mathcal{G}$-metric induced by* v, and the $\mathcal{G}$-inconsistency indicator map induced by d_v will be called *the inconsistency indicator map* induced by the $\mathcal{G}$-absolute value v.

Example 3.28. For $x \in \mathbb{R}_+$, we define $v(x) = 1 - \min\{x, x^{-1}\}$. Let us check that $v : \mathbb{R}_+ \to \mathbb{R}$ is a real absolute value on the standard multiplicative group $\mathbb{R}_+$ of positive real numbers (see Fig. 1).

Let $x, y \in \mathbb{R}_+$. Evidently, $0 \le v(x)$. Moreover,

$$v(x) = 0 \Leftrightarrow \min\{x, x^{-1}\} = 1 \Leftrightarrow x = 1.$$

The inequality $v(x \cdot y) \le v(x) + v(y)$ is equivalent to the following inequality (A):

$$\min\{x, x^{-1}\} + \min\{y, y^{-1}\} \le 1 + \min\{xy, (xy)^{-1}\}.$$

To prove (A), we shall consider the following cases:

(1) $xy \le x \le 1$. Then: $(A) \Leftrightarrow x + y \le 1 + xy \Leftrightarrow 0 \le (1 - x)(1 - y)$.
(2) $xy \le 1 \le x$. Then: $(A) \Leftrightarrow x^{-1} + y \le 1 + xy \Leftrightarrow 0 \le (x - 1)(1 + xy)$.
(3) $x \le xy \le 1$. Then: $(A) \Leftrightarrow x + y^{-1} \le 1 + xy \Leftrightarrow 0 \le (y - 1)(1 + xy)$.

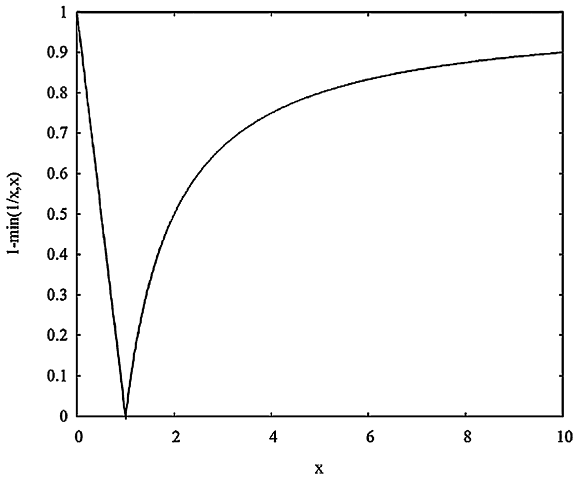

Fig. 1. A real absolute value on the standard multiplicative group of positive real numbers.

(4) $x \le 1 \le xy$. Then: $(A) \Leftrightarrow x + y^{-1} \le 1 + (xy)^{-1} \Leftrightarrow 0 \le (xy+1)(1-x)$.
(5) $1 \le xy \le x$. Then: $(A) \Leftrightarrow x^{-1} + y \le 1 + (xy)^{-1} \Leftrightarrow 0 \le (xy+1)(1-y)$.
(6) $1 \le x \le xy$. Then: $(A) \Leftrightarrow x^{-1} + y^{-1} \le 1 + (xy)^{-1} \Leftrightarrow 0 \le (1-x)(1-y)$.

This implies that (A) holds and, in consequence, v is a real absolute value. In view of Fact 3.26, the function d, defined by $d(x, y) = v(xy^{-1})$ for all $x, y \in \mathbb{R}_+$, is a metric on $\mathbb{R}_+$. Let T_d be the inconsistency indicator map induced by the real absolute value v on the group $\mathbb{R}_+$. Obviously, T_d is pairwise symmetric and bounded by 1. Furthermore, if we use the function $KI : \mathbb{R}^3_+ \to \mathbb{R}$, defined by $KI(x, y, z) = 1 - \min\{\frac{y}{xz}, \frac{zx}{y}\}$ for all $x, y, z \in \mathbb{R}_+$, we see that $KI = T_d$. Notice that KI was defined in [8]. The articles [1] and [11] analyze KI; however, they do not demonstrate that KI is induced by a real absolute value.

Example 3.29. Let $X = \mathbb{R}$ and $Y = (0; +\infty)$. Let $+$ and $\cdot$ denote the standard addition in X and, respectively, standard multiplication in Y. Suppose that a is a fixed positive real number. Obviously, the mapping $\Phi : X \ni x \mapsto a^x \in Y$ is an isomorphism of the group $\langle X, +\rangle$ onto the group $\mathbb{R}_+ = \langle Y, \cdot\rangle$. By Facts 3.4, 3.7 and 3.25, taken together with Example 3.28, we may define a $\mathcal{G}$-absolute value $w_a : X \to G$, a $\mathcal{G}$-metric $\rho_a : X^2 \to G$ and a $\mathcal{G}$-inconsistency indicator map $S_a : X^3 \to G$ by the formulas:

$$\begin{aligned} w_a(x) &= v(a^x) = 1 - \min\{a^x, a^{-x}\}, \\ \rho_a(x, y) &= d_v(a^x, a^y) = \max\{v(a^x, a^{-y}), v(a^y, a^{-x})\} \\ &= 1 - \min\{a^{x-y}, a^{y-x}\}, \\ S_a(x, y, z) &= \rho_a(x+z, y) = 1 - \min\{a^{x+z-y}, a^{y-x-z}\} \end{aligned}$$

for all $x, y, z \in X$. The indicator map S_a is also pairwise symmetric and bounded by 1. The function ρ_2 is illustrated by Fig. 2.

Remark 3.30. Usually, a mapping $T : X^3 \to Y$ is called *symmetric* if, for any $\langle x_1, x_2, x_3\rangle \in X^3$ and for any permutation σ of the set $\{1, 2, 3\}$, the equality

$$T(x_1, x_2, x_3) = T(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$$

holds. We will demonstrate how a $\mathcal{G}$-inconsistency indicator map can be symmetric for only very special groups.

Definition 3.31. Let T be a $\mathcal{G}$-inconsistency indicator map on a group X. Let $\mathcal{S}_3$ be the set of all permutations of the set $\{1, 2, 3\}$. The *full symmetrization* of T is the mapping $T^f : X^3 \to G$, defined by the formula:

$$T^f(x_1, x_2, x_3) = \max\{T(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) : \sigma \in \mathcal{S}_3\}$$

for all $x_1, x_2, x_3 \in X$.

Theorem 3.32. *Let T be a $\mathcal{G}$-inconsistency indicator map on a group X. Therefore, the following conditions hold:*

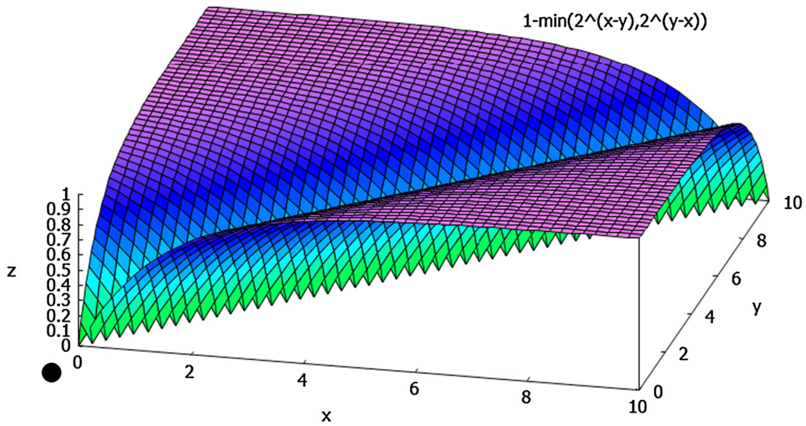

Fig. 2. $\mathcal{G}$-metric defined in Example 3.29 for $a = 2$.

(i) *if T^f is a $\mathcal{G}$-inconsistency indicator map on X, then $x^2 = 1$ for each $x \in X$;*
(ii) *if T is induced by a $\mathcal{G}$-absolute value on X, while $x^2 = 1$ for each $x \in X$, then $T^f = T$.*

Proof. Let d_T be the $\mathcal{G}$-metric induced by T. For $x, y \in X$, we define $\rho(x, y) = T^f(x, y, 1)$. It is easy to observe that

$$\rho(x, y) = \max\{d_T(x, y), d_T(xy, 1), d_T(yx, 1)\}.$$

Assume that T^f is a $\mathcal{G}$-inconsistency indicator map on X. Since, by Proposition 3.8, the function $\rho : X^2 \to G$ is a $\mathcal{G}$-metric on X, we have $\rho(x, x) = 1_G$, which implies that $d_T(xx, 1) = 1_G$. Hence $x^2 = 1$. This proves that (i) is satisfied.

To prove (ii), suppose that T is induced by a $\mathcal{G}$-absolute value v and that $x^2 = 1$ for each $x \in X$. Therefore, the group X is abelian. Therefore, we can notice that, for all $x, y \in X$, the following equalities hold: $d_T(xy, 1) = d_v(xy, 1) = v(xy) = d_T(x, y)$, so $\rho = d_v$ and $T^f = T$. □

Corollary 3.33. *Let T be a $\mathcal{G}$-inconsistency indicator map induced by a $\mathcal{G}$-absolute value on a group X. Therefore, T^f is a $\mathcal{G}$-inconsistency indicator map on X if and only if $x^2 = 1$ for each $x \in X$.*

Since every inconsistency indicator map on a group X is defined on X^3, while every metric on X is defined on X^2, it is reasonable to find such a generalized metric, defined on X^3, which is strictly relevant to a given inconsistency indicator map on X. It seems that generalized metrics introduced in [13] are most suitable to this aim. We modify Definition 3 of [13] as follows:

Definition 3.34. A $(3, \mathcal{G})$-*metric* on a set X is a function $g : X^3 \to \mathbb{R}$ which satisfies the following conditions:

(i) $g(x, y, z) = 1_G$ if $x = y = z \in X$;
(ii) $1_G < g(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
(iii) $g(x, x, y) \leq g(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
(iv) for every permutation σ of the set $\{1, 2, 3\}$ and for all $x_1, x_2, x_3 \in X$, the equality $g(x_1, x_2, x_3) = g(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$ holds;
(v) $g(x, y, z) \leq g(x, a, a) \odot g(a, y, z)$ for all $x, y, z, a \in X$.

Theorem 3.35. *Let X be a group. For a function $T : X^3 \to \mathcal{G}$ and for all $x, y, z \in X$, let*

$$g_T(x, y, z) = \max\{T(x, y, 1), T(x, z, 1), T(y, z, 1)\}.$$

Therefore, T is a $\mathcal{G}$-inconsistency indicator map on X if and only if the function $g_T : X^3 \to \mathbb{R}$ is a $(3, \mathcal{G})$-metric on X such that

$$T(x, y, z) = \max\{g(xz, y, y), g(xz, xz, y)\}$$

for all $x, y, z \in X$.

Proof. *Necessity.* Assume that T is a $\mathcal{G}$-inconsistency indicator map on X and let d_T be the $\mathcal{G}$-metric induced by T. Therefore, the function $g : X^3 \to G$, defined by

$$g(x, y, z) = \max\{d_T(x, y), d_T(y, z), d_T(x, z)\}$$

for all $x, y, z \in X$, is a $(3, \mathcal{G})$-metric on X (cf. [13]). It is easily seen that $g = g_T$ and that $\max\{g(xz, y, y), g(xz, xz, y)\} = d_T(xz, y) = T(x, y, z)$.

Sufficiency. Now, we assume that the function g_T is a $(3, \mathcal{G})$-metric on X such that $T(x, y, z) = \max\{g(xz, y, y), g(xz, xz, y)\}$ for all $x, y, z \in X$. Therefore, the function $\rho_g : X^2 \to G$, defined by

$$\rho_g(x, y) = \max\{g_T(x, y, y), g_T(x, x, y)\}$$

for all $x, y \in X$, is a $\mathcal{G}$-metric on X such that $\rho_g(xz, y) = T(x, y, z)$ for all $x, y, z \in X$. It follows from Theorem 3.13 that T is a $\mathcal{G}$-inconsistency indicator map on X and that ρ_g is the $\mathcal{G}$-metric induced by T. $\square$

4. *T*-inconsistency indicator

As in the previous section, we assume that $\mathcal{G} = \langle\langle G, \odot\rangle, \leq\rangle$ is an alo-group.

Definition 4.1. Let T be a $\mathcal{G}$-inconsistency indicator map on a group X. For a non-void finite subset C of X^3, let

$$\mathcal{I}_T[C] = \max\{T(x, y, z) : \langle x, y, z\rangle \in C\}.$$

We call $\mathcal{I}_T[C]$ the *T-inconsistency indicator of the set C*.

Definition 4.2. For $n \in \omega \setminus 1$, let $A = [a_{i,j}]$ be an $n \times n$ PC matrix over a group X and let T be a $\mathcal{G}$-inconsistency indicator map on X. Therefore, the *T-inconsistency indicator of the matrix A* is $\mathcal{I}_T[A]$ defined by

$$\mathcal{I}_T[A] = \max\{T(a_{i,k}, a_{i,j}, a_{k,j}) : i, j, k \in n\}.$$

In other words, $\mathcal{I}_T[A]$ is the $\mathcal{I}_T$-inconsistency indicator $\mathcal{I}_T[C(A)]$ of the set

$$C(A) = \{(a_{i,k}, a_{i,j}, a_{k,j}) : i, j, k \in n\} \subseteq X^3.$$

Remark 4.3. Let $A = [a_{i,j}]$ be a 3×3 PC matrix over G. In Definition 6.1 of [2], the consistency indicator $I_{\mathcal{G}}$ of A was defined as follows: $I_{\mathcal{G}}(A) = d_{\mathcal{G}}(a_{0,2}, a_{0,1} \odot a_{1,2})$ where $d_{\mathcal{G}}$ is the $\mathcal{G}$-metric induced by the $\mathcal{G}$-norm on G. In this case, $d_{\mathcal{G}}(a_{2,0}, a_{2,1} \odot a_{1,0}) = d_{\mathcal{G}}(a_{0,2}, a_{0,1} \odot a_{1,2})$. However, when, instead of $d_{\mathcal{G}}$, we consider the $\mathcal{G}$-metric d_T induced by a $\mathcal{G}$-inconsistency indicator map T on $\mathcal{G}$, it may happen that $d_T(a_{2,0}, a_{2,1} \odot a_{1,0}) \neq d_T(a_{0,2}, a_{0,1} \odot a_{1,2})$. This is partly why we do not define $\mathcal{I}_T[A]$ as $d_T(a_{0,2}, a_{0,1} \odot a_{1,2})$.

Example 4.4. Let us fix elements $a, b, c \in G$ such that $1_G < a < b < c$. Let $X_a = \{x \in G : 1_G \leq x\}$ and $X_b = G \setminus X_a$. For distinct $x, y \in X$, we put $\rho(x, x) = 1_G$, $\rho(x, y) = a$ if both x, y are elements of X_a, while $\rho(x, y) = b$ if both x, y are elements of X_b. Finally, $\rho(x, y) = c$ if either $x \in X_a$ and $y \in X_b$ or $x \in X_b$ and $y \in X_a$. Therefore, ρ is a $\mathcal{G}$-metric on G. Let T be the $\mathcal{G}$-inconsistency indicator map on G induced by ρ. Put $a_{i,i} = 1_G$ for each $i \in 3$, $a_{0,1} = a_{0,2} = a_{1,2} = a$ and $a_{1,0} = a_{2,0} = a_{2,1} = a^{-1}$. Therefore, $\rho(a_{0,1} \odot a_{1,2}, a_{0,2}) = T(a, a, a) = a$, while $\rho(a_{2,1} \odot a_{1,0}, a_{2,0}) = T(a^{-1}, a^{-1}, a^{-1}) = b$ and $\rho(a_{2,0} \odot a_{0,1}, a_{2,1}) = T(a^{-1}, a^{-1}, a) = c$. In consequence, $\mathcal{I}_T[A] = c$ and $\rho(a_{2,1} \odot a_{1,0}, a_{2,0}) \neq \mathcal{I}_T[A] \neq \rho(a_{0,1} \odot a_{1,2}, a_{0,2})$.

Corollary 4.5. *For $n \in \omega \setminus 3$, let $A = [a_{i,j}]$ be an $n \times n$ PC matrix over a group X and let T be a $\mathcal{G}$-inconsistency indicator map on X. Therefore:*

$$\mathcal{I}_T[A] = \mathcal{I}_{T^s}[A].$$

Proof. We deduce the above equation from the definition of T^s and from the equality $T^s(a_{i,k}, a_{i,j}, a_{k,j}) = T^s(a_{j,k}, a_{j,i}, a_{k,i})$. $\square$

Corollary 4.6. *Let $A = [a_{i,j}]$ be a 3×3 PC matrix over a group X and let T be a $\mathcal{G}$-inconsistency indicator map on X. Therefore,*

$$\mathcal{I}_T[A] = \max\{T^s(a_{0,1}, a_{0,2}, a_{1,2}), T^s(a_{1,0}, a_{1,2}, a_{0,2}), T^s(a_{0,2}, a_{0,1}, a_{2,1})\}.$$

Proof. It suffices to observe that the following equations hold:

$$T^s(a_{0,1}, a_{0,2}, a_{1,2}) = T^s(a_{2,1}, a_{2,0}, a_{1,0}),$$
$$T^s(a_{1,0}, a_{1,2}, a_{0,2}) = T^s(a_{2,0}, a_{2,1}, a_{0,1}),$$
$$T^s(a_{1,2}, a_{1,0}, a_{2,0}) = T^s(a_{0,2}, a_{0,1}, a_{1,2}). \quad \square$$

Example 4.4 demonstrates that, in general, the equality from Corollary 4.6 cannot be simplified; however, we can offer the following proposition.

Proposition 4.7. *Suppose that v is a $\mathcal{G}$-absolute value on an abelian group X, d_v is the $\mathcal{G}$-metric induced by v, while T is the $\mathcal{G}$ inconsistency indicator map on X induced by the $\mathcal{G}$-metric d_v. Let $A = [a_{i,j}]$ be a 3×3 PC matrix over X. Therefore, T is pairwise symmetric and $\mathcal{I}_T[A] = T(a_{0,1}, a_{0,2}, a_{1,2})$.*

Proof. We observe that

$$T(x, y, z) = d_v(xz, y) = \max\{v((xz) \cdot y^{-1}), v(yz^{-1}x^{-1})\} = T^s(x, y, z).$$

Let $a_{0,1} = x, a_{0,2} = y, a_{1,2} = z$. Since X is abelian, it is easily seen that $T(x, y, z) = T(x^{-1}, z, y) = T(y, x, z^{-1})$. We apply Corollary 4.6 to conclude the proof. □

Remark 4.8. In the light of Proposition 4.7, the consistency index from definition 6.2 of [2] is a particular case of our T-inconsistency indicator of a matrix.

Fact 4.9. *Let T_p be a $\mathcal{G}$-inconsistency indicator map on a group X_p for $p \in \{1, 2\}$. Suppose that $A_p = [a_{i,j}]$ is an $n \times n$ PC matrix over the group X_p for $p \in \{1, 2\}$. Let $X = X_1 \times X_2$, while*

$$T((x_1, x_2), (y_1, y_2), (z_1, z_2)) = \max\{T_1(x_1, y_1, z_1), T_2(x_2, y_2, z_2)\}$$

for all $x_p, y_p, z_p \in X_p$ and $p \in \{1, 2\}$. For all $i, j \in n$, let $c_{i,j} = (a_{i,j}, b_{i,j})$. Therefore, for the $n \times n$ PC matrix $C = [c_{i,j}]$ over X, we have

$$\mathcal{I}_T[C] = \max\{\mathcal{I}_{T_1}[A], \mathcal{I}_{T_2}[B]\}.$$

Example 4.10. For $n \in \omega \setminus 2$, let $A = [a_{i,j}]$ be an $n \times n$ PC matrix over the group $\mathbb{R}_+$. If T is the inconsistency indicator map on $\mathbb{R}_+$ induced by the real absolute value defined in Example 3.28, then $\mathcal{I}_T[A] = ii(A)$ where $ii(A)$ is the same as in Theorem 1 of [11].

In the literature on the theory of pairwise comparisons, the multiplicative PC matrices are often transformed into additive PC matrices by a logarithmic operation, probably used for the first time in [18]. Therefore, it is reasonable to apply Example 3.29 as follows.

Example 4.11. Suppose that a is a given positive real number, while $B = [b_{i,j}]$ is an $n \times n$ PC matrix over the group $\langle \mathbb{R}, + \rangle$. When S_a is the inconsistency indicator map defined in Example 3.29, then the S_a-inconsistency indicator of B is given by the following formula:

$$\begin{aligned}\mathcal{I}_{S_a}[B] &= \max\{S_a(b_{i,j}, b_{i,k}, b_{j,k}) : i, j, k \in n\} \\ &= \max\{1 - \min\{a^{b_{i,j}+b_{j,k}-b_{i,k}}, a^{b_{i,k}-b_{i,j}-b_{j,k}}\} : i, j, k \in n\}.\end{aligned}$$

The matrix B is additively consistent if and only if its elements satisfy equations $b_{i,j} = b_{i,k} + b_{k,j}$ for all $i, j, k \in n$. It is obvious that B is additively consistent if and only if $\mathcal{I}_{S_a}[B] = 0$.

Remark 4.12. Suppose that T is a $\mathcal{G}$-inconsistency indicator map on a group X and that A is an $n \times n$ PC matrix over X where $n \in \omega \setminus 2$. Therefore, A is not consistent if and only if $\mathcal{I}_T[A] > 1_G$. Since $\mathcal{I}_T[A] = 1_G$ holds only when A is consistent, for $\mathcal{I}_T[A]$, we prefer the term "T-inconsistency indicator" to the term "a consistency index".

5. Conclusions

This study generalizes some approaches to inconsistency indicators of reciprocal pairwise comparisons matrices given, for instance, in [2], [8] and [11].

For an abelian linearly ordered group $\mathcal{G}$, a new concept of a $\mathcal{G}$-inconsistency indicator map on a not necessarily abelian group X has been introduced and its basic properties have been investigated. The notions of a $\mathcal{G}$-metric and a $\mathcal{G}$-norm, given by in [2], have been generalized. Generalized metrics in [13] have also been investigated and connected with inconsistency indicator maps. Finally, using the inconsistency indicator maps, we have many distinct inconsistency indicators for pairwise comparisons matrices over groups. The inconsistency indicator, proposed by Koczkodaj for a single triad in 1993 in [8], as well as the consistency index independently defined by the formula of Definition 6.2 in [2] 16 years later, are special cases of the distance-based inconsistency indicator map introduced in Definition 3.6 of this study.

By an algebraic analysis, we have prepared general and more precise mathematical tools for pairwise comparisons methods. This work gives a group-theoretical perspective on problems of inconsistency in pairwise comparisons. Other properties and applications of the inconsistency indicator maps should be a subject of future research. In particular, the presented approach could be adjusted to accommodate uncertainty by using evidence theory (e.g. [15]). The proof of the inconsistency reduction process (based on the improvement of the worse triad) may be a challenging problem to solve for the proposed generalization. It was proved, only for the inconsistency proposed by Koczkodaj, in [8] with the recent Monte Carlo experimentation in [9]. The mathematical properties of the ratio scale (examined in [6] and [10]) should be revisited for the proposed generalization.

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