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# Computing a Consistent Approximation to a Generalized Pairwise Comparisons Matrix

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**Abstract**—This paper presents an algorithm for computing a consistent approximation to a generalized pairwise comparisons matrix (that is, without the reciprocity property or even 1s on the main diagonal). The algorithm is based on a logarithmic transformation of the generalized pairwise comparisons matrix into a linear space with the Euclidean metric. It uses both the row and (reciprocals of) column geometric means and is thus a generalization of the ordinary geometric means method. The resulting approximation is not only consistent, but also closest to the original matrix, i.e., deviates least from an expert's original judgments. The computational complexity of the algorithm is $O(n^2)$. © 1999 Elsevier Science Ltd. All rights reserved.

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## 1. BASICS OF PAIRWISE COMPARISONS

The method of pairwise comparisons (also known as paired comparisons) was introduced in embryonic form by Fechner (see [1]). It was made popular by Thurstone (see [2]) who extended it from binary to general choices. The introduction of hierarchical structures by Saaty in [3] was another improvement since it enabled the method to handle large numbers of criteria, for which the $O(n^2)$ complexity had been a major practical obstacle (larger groups can be split into smaller groups in a hierarchical structure). Only the essential concepts of the pairwise comparisons method are recalled here since this paper presents a generalization of the pairwise comparisons method presented in [4]. (URL http://www.laurentian.ca/www/math/wkocz/ref.html

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contains some additional references and a copy of *the Concluder* system which implements a consistency-driven pairwise comparisons method.)

An $n \times n$ pairwise comparisons matrix is defined as a square matrix $A = [a_{ij}]$ such that $a_{ij} > 0$ for $i, j = 1, \ldots, n$. Each $a_{ij}$ expresses a relative preference of stimulus (or criterion) $s_i$ over stimulus $s_j$ for $i, j = 1, \ldots, n$, represented by numerical weights (positive real numbers) $w_i$ and $w_j$, respectively. The quotients $a_{ij} = w_i/w_j$ form a pairwise comparisons matrix

$$A = \begin{vmatrix} 1 & a_{12} & \cdots & a_{1n} \\ \dfrac{1}{a_{12}} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{1}{a_{1n}} & \dfrac{1}{a_{2n}} & \cdots & 1 \end{vmatrix}.$$

A pairwise comparisons matrix $A$ is called *reciprocal* if $a_{ij} = 1/a_{ji}$ for $i, j = 1, \ldots, n$ (then automatically $a_{ii} = 1$ for $i = 1, \ldots, n$). A pairwise comparisons matrix $A$ is called *consistent* if $a_{ij} \cdot a_{jk} = a_{ik}$ holds for $i, j, k = 1, \ldots, n$ since $(w_i/w_j)(w_j/w_k)$ is naturally expected to be equal to $w_i/w_k$. In practice, comparing $s_i$ to $s_j$, $s_j$ to $s_k$, and $s_i$ to $s_k$ often results in inconsistency amongst the assessments in addition to their inaccuracy; however, the inconsistency may be computed and used to improve the accuracy. An improvement from about a 15% error to a 5% error was observed and verified statistically (see [5]). Saaty's Theorem (see [3]) states that for every $n \times n$ consistent matrix $A = [a_{ij}]$ there exist positive real numbers $w_1, \ldots, w_n$ (weights corresponding to stimuli $s_1, \ldots, s_n$) such that $a_{ij} = w_i/w_j$ for $i, j = 1, \ldots, n$. The weights $w_i$ are unique up to a multiplicative constant.

The challenge to the pairwise comparisons method comes from the lack of consistency of the pairwise comparisons matrices. Given an inconsistent $n \times n$ matrix $A$, the theory attempts to provide a consistent $n \times n$ matrix $A'$ which differs from matrix $A$ "as little as possible". A statistical experiment (see [6]) has shown that the accuracy of the weights does not depend strongly on the method used. In particular, the geometric means method (see [7,8]) produced results similar (to a high accuracy) to the eigenvector method (introduced in [3]) for the ten million cases tested. However, a strong relationship has been observed (see [6]) between accuracy and consistency. This is the main focus of the consistency-driven approach based on the triad-based inconsistency introduced in [9].

## 2. PAIRWISE COMPARISONS AND THE LINEARIZATION PROCESS

An algorithm for finding the nearest reciprocal and consistent matrix for a pairwise comparisons matrix which is not necessarily reciprocal is introduced. Relaxing the reciprocity assumption is an important step forward since according to [10], "as in virtually all previous papers, reciprocal $a_{ji} = 1/a_{ij}$ responses are assumed such that for $n$ comparison alternatives only $n(n-1)/2$ judgements are needed". The reciprocity condition has never been questioned since it is convenient for computing inverses. However, there is a problem when superfluous assessments are allowed. In practical applications, even comparing the same object to itself may not always yield 1 (e.g., blind testing of DNA samples, blind tasting of wines, etc.).

Suppose we somehow arrive at assessments $a_{ij} = 4$ and $a_{ji} = 1/3$. What shall we do with them? Replacing one value by the inverse of the other is not an acceptable solution. We may try to consider two different cases, one with $a_{ij} = 4$ and $a_{ji} = 1/4$, and the other with $a_{ij} = 3$ and $a_{ji} = 1/3$. This approach, however, has $O(2^n)$ complexity and even for a moderate $n = 32$ is computationally impracticable.

The proposed approximation algorithm provides a solution to the above problem. The problem of the best approximation of a given pairwise comparisons matrix $A = [a_{ij}]$ by a consistent matrix

is transformed into a corresponding problem of approximating a matrix $B = [\log a_{ij}]$ by a logarithmic image of a consistent matrix. The main benefit of this transformation (described in [7]) is that the logarithmically transformed images of consistent matrices form a linear subspace $L$ in $R^{n\times n}$. Each matrix in the subspace $L$ is called a triad $L$-consistent matrix $B = [b_{ij}]$ and satisfies the condition: $b_{ik} + b_{kj} = b_{ij}$ for $i, j, k = 1, \ldots, n$. It is much easier to work with linear spaces and to use the tools of linear algebra than to work in manifolds (topological or differential). Also, the notion of closeness of matrices (addressed in [7]) is preserved from one space to the other since the logarithmic transformation is homeomorphic (a one-to-one continuous mapping with a continuous inverse). Two matrices are thus close to each other in accordance with the Euclidean metric if their logarithmic images are also close with respect to the Euclidean metric. The approximation problem is thus reduced to the problem of finding the orthogonal projection of the matrix $B$ on $L$ since we opt for the least squares approximation in the space of logarithmic images of matrices.

## 3. THE APPROXIMATION ALGORITHM IN THE LINEAR SUBSPACE

The approximation algorithm presented here is based on the minimization of the Euclidean distance between the logarithmic image $B = [b_{ij}]$ of the given matrix and the set of logarithmically consistent matrices, i.e., of the function

$$d(B, B') = \sqrt{\sum_{i,j=1}^{n} \left(b_{ij} - b'_{ij}\right)^2}, \qquad \text{where } B, B' \in L.$$

We have to find a set of weights $(w_1, \ldots, w_n)$ which after the logarithmic transformation into the sequence $(x_1, \ldots, x_n)$ allows us to express the matrix $B'$ as $B' = [x_i - x_j]$ and, subsequently, the distance function $d$ in the form

$$d(B, B') = \sqrt{\sum_{i,j=1}^{n} \left(b_{ij} - (x_i - x_j)\right)^2}. \tag{1}$$

Note that after the transformation $b'_{ij} = x_i - x_j$, the consistency condition $b'_{ik} + b'_{kj} = b'_{ij}$ is automatically satisfied. Since $d$ is a differentiable function of the $n$ variables $x_i$, $i = 1, \ldots, n$, its minimum can be found by using standard calculus methods. The set of $n$ weights $w_i$, $i = 1, \ldots, n$ can be determined only up to a multiplicative factor and thus one extra condition is required to make them unique. Without loss of generality, we choose $w_n = 1$ (equivalent to dividing all the weights by $w_n$) since under a logarithmic transformation, this condition maps into $x_n = 0$. Thus, the problem of finding the nearest consistent reciprocal matrix is reduced to minimizing the function

$$f(B, x_1, \ldots, x_n) = \sum_{i,j=1}^{n} \left(b_{ij} - (x_i - x_j)\right)^2, \tag{2}$$

subject to the constraint $x_n = 0$. Since $f$ is a quadratic function, its minimum is guaranteed to exist. The method of Lagrange multipliers will be used since it preserves the symmetry of the problem. To this end, we form the function $u = f + \lambda x_n$ and differentiate with respect to $x_k$

$$\frac{\partial u}{\partial x_k} = \sum_{i,j=1}^{n} -2\left(b_{ij} - (x_i - x_j)\right)\left(\frac{\partial x_i}{\partial x_k} - \frac{\partial x_j}{\partial x_k}\right) + \lambda \frac{\partial x_n}{\partial x_k}, \qquad \text{for } k = 1, \ldots, n. \tag{3}$$

By Lagrange's method, we solve the equations $\frac{\partial u}{\partial x_k} = 0$ along with the constraint equation $x_n = 0$ for $x_i$, $i = 1, \ldots, n$ and the multiplier $\lambda$. Since $\frac{\partial x_i}{\partial x_k} = \delta_{ik}$, where $\delta_{ik}$ is the Kronecker

delta symbol, the above equations can be simplified by carrying out one of the summations. This leads to

$$\begin{aligned}\frac{\partial u}{\partial x_k} &= -2\sum_{j=1}^{n}(b_{kj}-(x_k-x_j))+2\sum_{i=1}^{n}(b_{ik}-(x_i-x_k))+\lambda\delta_{nk}\\ &= -2R_k+2C_k+4nx_k-4X+\lambda\delta_{nk}, \qquad \text{for } k=1,\ldots,n,\end{aligned} \tag{4}$$

where

$$X=\sum_{i=1}^{n}x_i,$$

$$R_k=\sum_{i=1}^{n}b_{ki},$$

$$C_k=\sum_{i=1}^{n}b_{ik}.$$

Setting the derivatives equal to zero yields

$$x_k=\frac{R_k-C_k}{2n}+\frac{1}{n}X-\frac{\lambda}{4n}\delta_{nk}, \qquad \text{for } k=1,\ldots,n, \tag{5}$$

together with the constraint

$$x_n=0.$$

By forming the sum

$$\sum_{k=1}^{n}x_k=\frac{\sum_{k=1}^{n}R_k-\sum_{k=1}^{n}C_k}{2n}+\frac{1}{n}\sum_{k=1}^{n}X-\frac{\lambda}{4n}, \tag{6}$$

and using the fact that

$$\sum_{k=1}^{n}R_k=\sum_{k=1}^{n}C_k, \tag{7}$$

we get $\lambda=0$. Hence

$$x_k=\frac{R_k-C_k}{2n}+\frac{1}{n}X, \qquad \text{for } k=1,\ldots,n, \tag{8}$$

and thus, the general formula for the entries of the matrix $B'$ is

$$b'_{ij}=x_i-x_j=\frac{1}{2}\left(\frac{R_i-C_i}{n}-\frac{R_j-C_j}{n}\right), \qquad \text{for } i,j=1,\ldots,n. \tag{9}$$

In summary, the following algorithm RCGM (Row/Column Geometric Means) transforms a generalized pairwise comparisons matrix into a consistent reciprocal matrix.

ALGORITHM RCGM

- *Input*: a pairwise comparisons matrix $A$ which does not need to be reciprocal nor have 1s on the main diagonal; however, all entries must be positive numbers.
- *Output*: a matrix $A'$, the fully consistent reciprocal approximation to $A$ with respect to the Euclidean metric.

Step 1. Compute $B=[b_{ij}]$ with elements $b_{ij}=\log a_{ij}$ for $i,j=1,\ldots,n$.
Step 2. Compute $R_i$, the row sums of $B$ for $i=1,\ldots,n$.
Step 3. Compute $C_i$, the column sums of $B$ for $i=1,\ldots,n$.
Step 4. Compute $B'=[b'_{ij}]$ with element defined by formula (9).
Step 5. Compute $A'=[a'_{ij}]$ with elements defined by formula (10),

$$a'_{ij}=\exp\left(b'_{ij}\right), \qquad \text{for } i,j=1,\ldots,n. \tag{10}$$

END ALGORITHM

Alternatively, by substituting the values of $b'_{ij}$ from formula (9) into formula (10) and performing some straightforward algebra, the matrix $A'$ can be computed directly in terms of the geometric means of the rows and columns of matrix $A$. The final result is

$$a'_{ij} = \sqrt{\frac{R_i^*}{R_j^*} \cdot \frac{C_j^*}{C_i^*}}, \qquad \text{for } i, j = 1, \ldots, n, \tag{11}$$

where

$$R_i^* = \sqrt[n]{\prod_{j=1}^{n} a_{ij}}, \qquad \text{for } i = 1, \ldots, n$$

and

$$C_i^* = \sqrt[n]{\prod_{j=1}^{n} a_{ji}}, \qquad \text{for } i = 1, \ldots, n.$$

It is worthwhile noting that the values $a'_{ij}$ in formula (11) may be interpreted as the geometric means of corresponding elements in two matrices one of which is constructed from the geometric means of the rows of $A$, and the other from the geometric means of the columns of $A$. For reciprocal matrices, for which $C_i^* = 1/R_i^*$, formula (11) reduces to

$$a_{ij} = \frac{R_i^*}{R_j^*} = \frac{C_j^*}{C_i^*}. \tag{12}$$

Formula (12) is the result for the geometric means method applied to rows or to columns. Thus, the result in formula (11) is a generalization of the geometric means method. Rather than using either the row geometric means or (the reciprocals of) the column geometric means individually, we use their geometric mean. It should not be surprising then that an improved accuracy is seen in the example considered in the next section.

The computational complexity of each step of algorithm RCGM is $O(n^2)$, and thus, the complexity of the entire algorithm is $O(n^2)$. (The space complexity is also $O(n^2)$.)

## 4. A NUMERICAL EXAMPLE

Algorithm RCGM has been tested on many pairwise comparisons matrices and the following results are typical of those obtained. The pairwise comparisons matrix $A$ has been randomly selected (the inconsistency index was used to discard very inconsistent cases; this process is described in [6]),

$$A = \begin{bmatrix} 1.2 & 2 & 0.5 & 3 \\ 0.4 & 0.9 & 0.25 & 1.5 \\ 1.5 & 3 & 1 & 5 \\ 0.25 & 0.5 & 0.33 & 1.1 \end{bmatrix}.$$

A consistent approximation to the matrix was computed using algorithm RCGM and the results were compared with those obtained using the other known methods. Matrix $A$ is evidently not reciprocal (e.g., elements $a_{12}$ and $a_{21}$ are not inverses of each other) nor does the main diagonal consist entirely of 1s. The values of $d(A, A')$ for the selected reconstruction algorithms are shown in Table 1.

The distances from the given matrix to the matrices reconstructed from the Eigenvector (EV), the Row Geometric Means (RGM), and the Column Geometric Means (CGM) are all greater than the distance to the matrix reconstructed by the algorithm based on a combination of the

Table 1. Distances between the approximation matrices and the given matrix.

| Algorithm | Weights | Approximation Matrix $A'$ | $d(A, A')$ |
|---|---|---|---|
| EV | $\begin{bmatrix} 2.851 \\ 1.248 \\ 4.471 \\ 1.000 \end{bmatrix}$ | $\begin{bmatrix} 1 & 2.285 & 0.638 & 2.851 \\ 0.438 & 1 & 0.279 & 1.248 \\ 1.568 & 3.583 & 1 & 4.471 \\ 0.351 & 0.801 & 0.224 & 1 \end{bmatrix}$ | 0.993 |
| RGM | $\begin{bmatrix} 2.984 \\ 1.313 \\ 4.719 \\ 1.000 \end{bmatrix}$ | $\begin{bmatrix} 1 & 2.272 & 0.632 & 2.984 \\ 0.440 & 1 & 0.278 & 1.313 \\ 1.581 & 3.593 & 1 & 4.719 \\ 0.335 & 0.761 & 0.212 & 1 \end{bmatrix}$ | 0.846 |
| CGM | $\begin{bmatrix} 3.424 \\ 1.740 \\ 4.949 \\ 1.000 \end{bmatrix}$ | $\begin{bmatrix} 1 & 1.968 & 0.692 & 3.424 \\ 0.508 & 1 & 0.352 & 1.740 \\ 1.445 & 2.844 & 1 & 4.949 \\ 0.292 & 0.575 & 0.202 & 1 \end{bmatrix}$ | 0.641 |
| RCGM | $\begin{bmatrix} 3.197 \\ 1.512 \\ 4.833 \\ 1.000 \end{bmatrix}$ | $\begin{bmatrix} 1 & 2.115 & 0.662 & 3.197 \\ 0.473 & 1 & 0.313 & 1.512 \\ 1.512 & 3.197 & 1 & 4.833 \\ 0.313 & 0.662 & 0.207 & 1 \end{bmatrix}$ | 0.509 |

Row and Column Geometric Means (RCGM). The eigenvector corresponding to the largest eigenvalue 3.9119 was computed as $[0.537, 0.23504, 0.84222, 0.18837]$ and scaled (as were the weights obtained by the other methods) to have the last coordinate $w_n$ set to 1 as was done in Section 3.

## 5. CONCLUSIONS

The algorithm RCGM presented here for computing a consistent approximation to a generalized pairwise comparisons matrix (that is, without the reciprocity property or even 1s on the main diagonal) is another step forward for practical applications. It accommodates imprecise data and is suitable for managing uncertainty. By relaxing the reciprocity assumption we allow the user a more flexible data acquisition. Superfluous data are no longer a data entry nuisance, but instead are incorporated to produce an improved estimate of the corresponding consistent approximation. Second, our algorithm produces a consistent approximation for any matrix with positive elements, i.e., even in those situations where the eigenvector method is inapplicable (in general, the largest absolute eigenvalue of a nonreciprocal matrix may be complex). This generalization, to matrices with arbitrary positive elements, can be considered the last possible from an application viewpoint. (An interpretation for comparisons involving any negative numbers, e.g., "$A$ is $-2.5$ better than $B$" remains to be discovered.) And finally, algorithm RCGM produces a matrix which is not only consistent but is also closest to the original matrix. As stated by one of the referees: "either we respect experts' judgments or not".

The new algorithm is efficient. It is of complexity $O(n^2)$ and nothing better is possible, bearing in mind that the complexity of a total search involving matrices of size $n \times n$ cannot be better than $O(n^2)$.

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